Additional Questions

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Week 11

L'Hôpital's Rule

1. Let $f(x) = 5$ and $g(x) = (x+2)^2(x^2+4x+1)$.
   (a) Find the limit as $x \to -2$ of $\frac{f(x)}{g(x)}$.
   (b) Find the limit as $x \to -2$ of $\frac{f'(x)}{g'(x)}$.

   This shows that L'Hôpital's rule can only be applied for the specific indeterminate forms.

   Solution

   (a) Since $g(x) = (x+2)^2(x^2+4x+1) = x^4+8x^3+21x^2+20x+4 < 5(x-2)^6$ for all $x \in \mathbb{R} - \{-2\}$, $\frac{f(x)}{g(x)} > \frac{1}{(x-2)^6}$. Since $\frac{1}{(x-2)^6} \to \infty$ as $x \to -2$, $\frac{f(x)}{g(x)} \to \infty$ as $x \to -2$.
   (b) By definition, $f'(x) = 0$ and $g'(x) = 4x^3+24x^2+42x+20$, so $\frac{f'(x)}{g'(x)} = 0$. Hence, $\frac{f'(x)}{g'(x)} \to 0$ as $x \to -2$.

2. Use L'Hôpital's rule to evaluate the following limits:
   (a) $\lim_{x\to -4} \frac{\sin(\pi x)}{x^2-16}$
   (b) $\lim_{x\to\infty} \frac{\ln(3x)}{x^2}$
   (c) $\lim_{x\to 0} \frac{\sin(2x)+7x^2-2x}{x^2(x-1)^2}$

   Solution

   (a) Let $f(x) = \sin(\pi x)$ and $g(x) = x^2 - 16$, so $f(-4) = g(-4) = 0$, so L'Hôpital's rule can be applied. Since $f'(x) = \pi\cos(\pi x)$ and $g'(x) = 2x$,

$$\begin{aligned}\lim_{x\to -4} \frac{\sin(\pi x)}{x^2-16} &= \lim_{x\to -4} \frac{\pi\cos(\pi x)}{2x} \\ &= \frac{\lim_{x\to -4} \pi\cos(\pi x)}{\lim_{x\to -4} 2x} \\ &= \frac{\pi\cos(-4\pi)}{-8} \\ &= -\frac{\pi}{8}\end{aligned}$$

(b) Let $f(x) = \ln(3x)$ and $g(x) = x^2$, so $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$, so L'Hôpital's rule can be applied. Since $f'(x) = \frac{1}{x}$ and $g'(x) = 2x$,

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(3x)}{x^2} &= \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} \\ &= \lim_{x\to\infty} \frac{1}{2x^2} \\ &= \frac{1}{2} \lim_{x\to\infty} \frac{1}{x^2} \\ &= 0\end{aligned}$$

(c) Let $f(x) = \sin(2x) + 7x^2 - 2x$ and $g(x) = x^2(x-1)^2$, so $f(0) = g(0) = 0$, so L'Hôpital's rule can be applied. Since $f'(x) = 2\cos(2x) + 14x - 2$ and $g'(x) = 2x(x-1)^2 + 2x^2(x-1)$, $f'(0) = g'(0) = 0$, so L'Hôpital's rule can be applied again. Since $f''(x) = -4\sin(2x) + 14$ and $g''(x) = 2(x-1)^2 + 4x(x-1) + 4x(x-1) + 2x^2 = 2(x-1)(5x-1) + 2x^2$,

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(2x) + 7x^2 - 2x}{x^2(x-1)^2} &= \lim_{x\to 0} \frac{-4\sin(2x) + 14}{2(x-1)(5x-1) + 2x^2} \\ &= \frac{\lim_{x\to 0}(-4\sin(2x) + 14)}{\lim_{x\to 0}(2(x-1)(5x-1) + 2x^2)} \\ &= \frac{14}{2} \\ &= 7\end{aligned}$$

3. Use L'Hôpital's rule to evaluate the limit $\lim_{x\to\infty} x\ln(1 + \frac{3}{x})$.

Solution Since

$$\lim_{x\to\infty} x\ln\left(1 + \frac{3}{x}\right) = \lim_{x\to\infty} \frac{\ln(1 + \frac{3}{x})}{\frac{1}{x}}$$

and if $t = \frac{1}{x}$, then $x \to \infty$ if and only if $t \to 0^+$, it follows that

$$\lim_{x\to\infty} x\ln\left(1 + \frac{3}{x}\right) = \lim_{t\to 0^+} \frac{\ln(1 + 3t)}{t}$$

Let $f(t) = \ln(1 + 3t)$ and $g(t) = t$, so $f(0) = g(0) = 0$, so L'Hôpital's rule can be applied. Since $f'(x) = \frac{3}{1+3t}$ and $g'(t) = 1$,

$$\begin{aligned}\lim_{x\to\infty} x\ln\left(1 + \frac{3}{x}\right) &= \lim_{t\to 0^+} \frac{\ln(1 + 3t)}{t} \\ &= \lim_{t\to 0^+} \frac{3}{1 + 3t} \\ &= \frac{3}{\lim_{t\to 0^+}(1 + 3t)} \\ &= 3\end{aligned}$$

4. Use L'Hôpital's rule to evaluate the limit $\lim_{x\to\infty}(e^x + x)^{\frac{1}{x}}$.

Solution Consider

$$\lim_{x\to\infty} \ln((e^x + x)^{\frac{1}{x}}) = \lim_{x\to\infty} \frac{\ln(e^x + x)}{x}$$

Let $f(x) = \ln(e^x + x)$ and $g(x) = x$, so $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$, so L'Hôpital's rule can be applied. Since $f'(x) = \frac{e^x+1}{e^x+x}$ and $g'(x) = 1$,

$$\lim_{x\to\infty} \ln((e^x + x)^{\frac{1}{x}}) = \lim_{x\to\infty} \frac{e^x + 1}{e^x + x}$$

Let $h(x) = e^x + 1$ and $k(x) = e^x + x$, so $h(x) \to \infty$ and $k(x) \to \infty$ as $x \to \infty$, so L'Hôpital's rule can be applied. Since $h'(x) = e^x$ and $k'(x) = e^x + 1$,

$$\begin{aligned} \lim_{x\to\infty} \ln((e^x + x)^{\frac{1}{x}}) &= \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \\ &= \lim_{x\to\infty} \frac{e^x}{e^x + 1} \\ &= \lim_{x\to\infty} \frac{1}{1 + e^{-x}} \\ &= 1 \end{aligned}$$

It follows that $\lim_{x\to\infty}(e^x + x)^{\frac{1}{x}} = e^{\lim_{x\to\infty} \ln((e^x+x)^{\frac{1}{x}})} = e$.

Improper Integrals

1. Determine if each of the integrals
   (a) $\int_0^\infty (1 + 2x)e^{-x}\,\mathrm{d}x$
   (b) $\int_{-\infty}^0 (1 + 2x)e^{-x}\,\mathrm{d}x$

   converges or diverges, and find its value if it converges.

Solution Evaluating the indefinite integral, let $u = 1 + 2x$ and $v = e^{-x}$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and $\int v\,\mathrm{d}x = -e^x$. Using integration by parts,

$$\begin{aligned} \int uv\,\mathrm{d}x &= u \int v\,\mathrm{d}x - \int \frac{\mathrm{d}u}{\mathrm{d}x} \left(\int v\,\mathrm{d}x \right) \mathrm{d}x \\ \implies \int (1 + 2x)e^{-x}\,\mathrm{d}x &= -(1 + 2x)e^{-x} + 2 \int e^{-x}\,\mathrm{d}x \\ &= -(1 + 2x)e^{-x} - 2e^{-x} + C \\ &= -(3 + 2x)e^{-x} + C \end{aligned}$$

where $C \in \mathbb{R}$ is the constant of integration.

(a) By definition,

$$\begin{aligned} \int_0^\infty (1 + 2x)e^{-x}\,\mathrm{d}x &= \lim_{t\to\infty} \int_0^t (1 + 2x)e^{-x}\,\mathrm{d}x \\ &= \lim_{t\to\infty} \left[-(3 + 2x)e^{-x} \right]_0^t \\ &= \lim_{t\to\infty} (-(3 + 2t)e^{-t} + 3) \\ &= \lim_{t\to\infty} \left(\frac{3e^t - 3 - 2t}{e^t} \right) \end{aligned}$$

Using L'Hôpital's rule,

$$\begin{aligned}\int_0^{\infty}(1+2x)e^{-x}\,\mathrm{d}x &= \lim_{t\to\infty}\left(\frac{3e^t-3-2t}{e^t}\right)\\ &= \lim_{t\to\infty}\left(\frac{3e^t-2}{e^t}\right)\\ &= \lim_{t\to\infty}(3-2e^{-t})\\ &= 3\end{aligned}$$

(b) By definition,

$$\begin{aligned}\int_{-\infty}^{0}(1+2x)e^{-x}\,\mathrm{d}x &= \lim_{t\to-\infty}\int_t^0(1+2x)e^{-x}\,\mathrm{d}x\\ &= \lim_{t\to-\infty}\left[-(3+2x)e^{-x}\right]_t^0\\ &= \lim_{t\to-\infty}(-3+(3+2t)e^{-t})\\ &= \lim_{t\to-\infty}\left(\frac{3+2t-3e^t}{e^t}\right)\end{aligned}$$

Since $\lim_{t\to-\infty}(\frac{3+2t-3e^t}{e^t})=-\infty$, the integral diverges.

2. Find the area of the region between the curve $y=\frac{7}{x^2}$ and the x-axis, bounded by $x=1$ on the left.

Solution Since $\frac{7}{x^2}>0$ for all $x\in[1,\infty)$, this area is given by

$$\begin{aligned}\left|\int_1^{\infty}\frac{7}{x^2}\,\mathrm{d}x\right| &= \int_1^{\infty}\frac{7}{x^2}\,\mathrm{d}x\\ &= \lim_{t\to\infty}\int_1^t\frac{7}{x^2}\,\mathrm{d}x\\ &= \lim_{t\to\infty}\left[-\frac{7}{x}\right]_1^t\\ &= \lim_{t\to\infty}(-7t^{-1}+7)\\ &= 7\end{aligned}$$

3. Find the area of the region between the curve $y=-\frac{1}{\sqrt{3-x}}$ and the x-axis, bounded by $x=0$ and $x=3$.

Solution Since $-\frac{1}{\sqrt{3-x}}<0$ for all $x\in[0,3)$, this area is given by

$$\begin{aligned}\left|\int_0^3-\frac{1}{\sqrt{3-x}}\,\mathrm{d}x\right| &= \int_0^3\frac{1}{\sqrt{3-x}}\,\mathrm{d}x\\ &= \lim_{t\to3}\int_0^t\frac{1}{\sqrt{3-x}}\,\mathrm{d}x\\ &= \lim_{t\to3}\left[-2\sqrt{3-x}\right]_0^t\\ &= \lim_{t\to3}(-2\sqrt{3-t}+2\sqrt{3})\\ &= 2\sqrt{3}\end{aligned}$$

Differential Equations

1. Use an integrating factor to find the general solution to the first order differential equation $\frac{dx}{dt} + \frac{x}{t} + te^{-t} = 0$ for $t > 0$.

   Solution Since $\frac{dx}{dt} + \frac{x}{t} + te^{-t} = 0$, $\frac{dx}{dt} + \frac{x}{t} = -te^{-t}$. Let

$$\begin{aligned} u(t) &= e^{\int \frac{1}{t}\, dt} \\ &= e^{\ln|t|} \\ &= t \end{aligned}$$

   Then,

$$\begin{aligned} u(t)\frac{dx}{dt} + u(t)\frac{x}{t} &= -te^{-t}u(t) \\ \implies t\frac{dx}{dt} + x &= -t^2e^{-t} \\ \implies \frac{d}{dt}(tx) &= -t^2e^{-t} \\ \implies tx &= \int -t^2e^{-t}\, dt \end{aligned}$$

   Let $w = -t^2$ and $y = e^{-t}$, so $\frac{dw}{dt} = -2t$ and $\int y\, dt = -e^{-t}$. Using integration by parts,

$$\begin{aligned} \int wy\, dt &= w\int y\, dt - \int \frac{dw}{dt}\left(\int y\, dt\right) dt \\ \implies \int -t^2e^{-t}\, dt &= t^2e^{-t} - 2\int te^{-t}\, dt \end{aligned}$$

   Let $z = t$, so $\frac{dz}{dt} = 1$. Using integration by parts again,

$$\begin{aligned} \int zy\, dt &= z\int y\, dt - \int \frac{dz}{dt}\left(\int y\, dt\right) dt \\ \implies \int te^{-t}\, dt &= -te^{-t} + \int e^{-t}\, dt \\ &= -te^{-t} - e^{-t} \end{aligned}$$

   Hence,

$$\begin{aligned} tx &= \int -t^2e^{-t}\, dt \\ &= t^2e^{-t} - 2\int te^{-t}\, dt \\ &= t^2e^{-t} - 2(-te^{-t} - e^{-t}) + C \\ &= (t^2 + 2t + 2)e^{-t} + C \end{aligned}$$

   and so the general solution is $x(t) = (t + 2 + 2t^{-1})e^{-t} + Ct^{-1}$ for some constant $C \in \mathbb{R}$.

2. A company opened a bank account with £100,000 at the start of 2010, which accrues 5% compound interest continuously. They withdraw £4,000 each year.

   (a) Construct a first order differential equation describing the rate of change of the account balance.

   (b) Find the general solution describing the account balance in terms of the time in years.

   (c) How much was in the bank account at the start of 2020?

   (d) In what year will the account balance reach £150,000?

Solution

(a) Let $M(t)$ be the account balance at time t. The interest added to the account each year is $0.05M(t)$ and the amount removed from the account each year is 4000, so

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0.05M(t) - 4000$$

(b) This is a separable equation with

$$\begin{aligned} \frac{1}{0.05M(t) - 4000}\frac{\mathrm{d}M}{\mathrm{d}t} &= 1 \\ \implies \frac{20}{M(t) - 80000}\frac{\mathrm{d}M}{\mathrm{d}t} &= \frac{1}{20} \end{aligned}$$

so it can be solved by integrating both sides. Hence,

$$\begin{aligned} \int \frac{1}{M(t) - 80000}\,\mathrm{d}M &= \int \frac{1}{20}\,\mathrm{d}t \\ \ln|M(t) - 80000| &= \frac{t}{20} + C \end{aligned}$$

for some constant $C \in \mathbb{R}$. Since $M(0) = 100000$, $C = \ln(20000)$. To rearrange to find $M(t)$,

$$\begin{aligned} \ln|M(t) - 80000| &= \frac{t}{20} + \ln(20000) \\ \implies e^{\ln|M(t)-80000|} &= e^{\frac{t}{20}+\ln(20000)} \\ \implies M(t) - 80000 &= 20000e^{\frac{t}{20}} \\ \implies M(t) &= 80000 + 20000e^{\frac{t}{20}} \end{aligned}$$

(c) Since $t = 0$ is the start of 2010, the start of 2020 is $t = 10$. Hence,

$$\begin{aligned} M(10) &= 80000 + 20000e^{\frac{1}{2}} \\ &= 112974.43 \qquad \text{(2 decimal places)} \end{aligned}$$

so at the start of 2020, there was £112,974.43 in the account.

(d) Let $M(t) = 150000$, so

$$\begin{aligned} 80000 + 20000e^{\frac{t}{20}} &= 150000 \\ \implies 8 + 2e^{\frac{t}{20}} &= 15 \\ \implies 2e^{\frac{t}{20}} &= 7 \\ \implies e^{\frac{t}{20}} &= 3.5 \\ \implies t &= 20 \times \ln(3.5) \\ &= 25.0553... \end{aligned}$$

Hence, the account balance will reach £150,000 in 2035.

3. Show that $x^2 - x^{-1}$ is a solution to the second order differential equation $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y$.

Solution To prove this, we need to show that $x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2 - x^{-1}) = 2(x^2 - x^{-1})$. Since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x^2 - x^{-1}) &= 2x + x^{-2} \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2 - x^{-1}) &= 2 - 2x^{-3} \end{aligned}$$

it follows that

$$\begin{aligned} x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2 - x^{-1}) &= x^2(2 - 2x^{-3}) \\ &= 2x^2 - 2x^{-1} \\ &= 2(x^2 - x^{-1}) \end{aligned}$$

so $x^2 - x^{-1}$ is a solution.

4. Suppose that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = f(x)$ for some function $f : \mathbb{R} \to \mathbb{R}$, and $\sin(x) + x^2$ is a solution to the differential equation. Find an expression for $f(x)$.

Solution Since $\sin(x) + x^2$ is a solution, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sin(x) + x^2) + (\sin(x) + x^2) = f(x)$. Since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\sin(x) + x^2) &= \cos(x) + 2x \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\sin(x) + x^2) &= -\sin(x) + 2 \end{aligned}$$

it follows that $f(x) = (-\sin(x) + 2) + (\sin(x) + x^2) = x^2 + 2$.

5. Consider the second order differential equation $ay''(x) + by'(x) + cy(x) = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Show that if e^{mx} and e^{nx} are two solutions to the differential equation, then $e^{mx} + e^{nx}$ is a solution to the differential equation.

Solution To prove this, we need to show that

$$a\frac{\mathrm{d}^2}{\mathrm{d}x^2}(e^{mx} + e^{nx}) + b\frac{\mathrm{d}}{\mathrm{d}x}(e^{mx} + e^{nx}) + c(e^{mx} + e^{nx}) = 0$$

Since e^{mx} and e^{nx} are both solutions, we know that

$$\begin{aligned} a\frac{\mathrm{d}^2}{\mathrm{d}x^2}(e^{mx}) + b\frac{\mathrm{d}}{\mathrm{d}x}(e^{mx}) + ce^{mx} &= 0 \\ \implies am^2e^{mx} + bme^{mx} + ce^{mx} &= 0 \\ a\frac{\mathrm{d}^2}{\mathrm{d}x^2}(e^{nx}) + b\frac{\mathrm{d}}{\mathrm{d}x}(e^{nx}) + ce^{nx} &= 0 \\ \implies an^2e^{nx} + bne^{nx} + ce^{nx} &= 0 \end{aligned}$$

Hence, it follows that

$$\begin{aligned} a\frac{\mathrm{d}^2}{\mathrm{d}x^2}(e^{mx} + e^{nx}) + b\frac{\mathrm{d}}{\mathrm{d}x}(e^{mx} + e^{nx}) + c(e^{mx} + e^{nx}) &= a(m^2e^{mx} + n^2e^{nx}) \\ &\quad + b(me^{mx} + ne^{nx}) + c(e^{mx} + e^{nx}) \\ &= (am^2e^{mx} + bme^{mx} + ce^{mx}) \\ &\quad + (an^2e^{nx} + bne^{nx} + ce^{nx}) \\ &= 0 \end{aligned}$$

so $e^{mx} + e^{nx}$ is a solution.