Set Theory

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Week 3

Class Content

Question 1

List the elements of the following sets:

1. $A = \{x \in \mathbb{N} : x^2 = 4\}$
2. $B = \{x \in \mathbb{Z} : x^2 = 4\}$
3. $C = \{x \in \mathbb{N} : -1 \leq x \leq 5\}$
4. $D = \{5x : x \in \mathbb{Z}, -3 < x < 2\}$
5. $E = \{x \in \mathbb{Q} : x^2 = 2\}$
6. $F = \{x \in \mathbb{R} : x^2 = 2\}$
7. $G = \{\sqrt{x} : x \in \mathbb{N}, x < 5\}$

Solution

1. $A = \{2\}$
2. $B = \{-2, 2\}$
3. $C = \{1, 2, 3, 4, 5\}$
4. $D = \{-10, -5, 0, 5\}$
5. $E = \varnothing$
6. $F = \{-\sqrt{2}, \sqrt{2}\}$
7. $G = \{1, \sqrt{2}, \sqrt{3}, 2\}$

Definition. *The* **intersection** *of a set A and a set B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$.*

Definition. *The* **union** *of a set A and a set B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.*

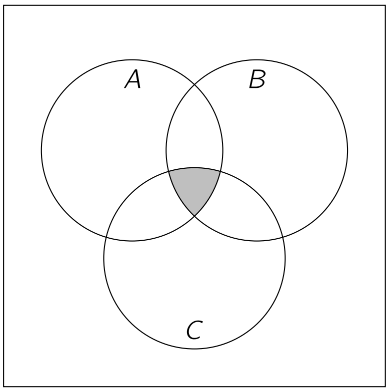
Definition. *The* **complement** *of a set A relative to a set B is the set $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.*

Question 2

Let A, B and C be sets. Draw a Venn diagram showing the following sets:

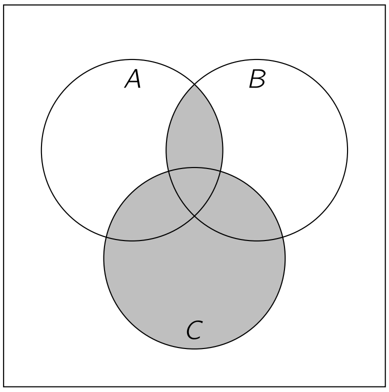
1. $(A \cap B) \cap C$

Solution



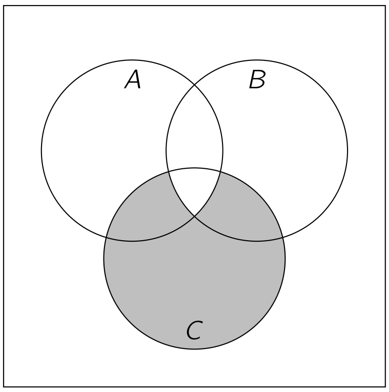
2. $(A \cap B) \cup C$

Solution



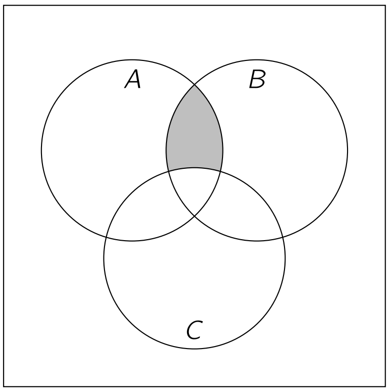
3. $C \setminus (A \cap B)$

Solution



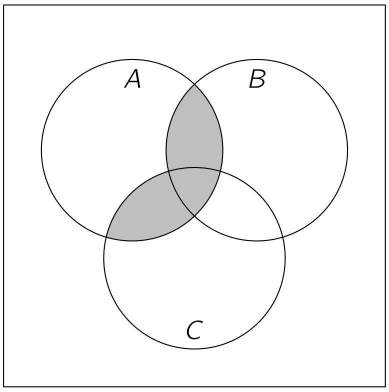
4. $(A \cap B) \setminus C$

Solution



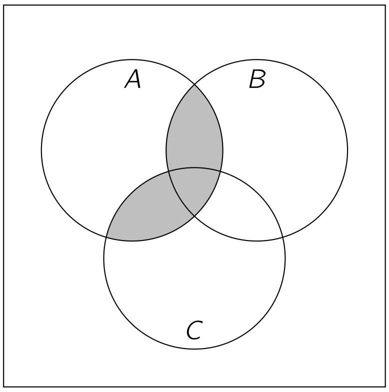
5. $A \cap (B \cup C)$

Solution



6. $(A \cap B) \cup (A \cap C)$

Solution



Question 3

Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for any sets A, B and C (the first distributive law).

Solution To prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Suppose that $x \in A \cap (B \cup C)$. By definition of the intersection, $x \in A$ and $x \in B \cup C$.

- If $x \in B$, then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.
- If $x \notin B$, then $x \in C$. This implies that $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$.

Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Suppose that $x \in (A \cap B) \cup (A \cap C)$. By definition of the union, $x \in A \cap B$ or $x \in A \cap C$, so $x \in A$ by definition of the intersection.

- If $x \in B$, then $x \in B \cup C$, so $x \in A \cap (B \cup C)$.
- If $x \notin B$, then $x \in C$. This implies that $x \in B \cup C$, so $x \in A \cap (B \cup C)$.

Therefore, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, and so $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Additional Questions

1. Write each of the following sets in set-builder notation (describing the set by giving a property that the elements must satisfy):

   (a) $\{-1, -2, -3, \dots\}$
   (b) $\{1, 3, 5, 7\}$
   (c) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$
   (d) $(0, 1) \cap \mathbb{Q}$
   (e) $[-10, 10] \cap \mathbb{N}$

   Can you think of another way of writing them?

   Solution

   (a) $\{n \in \mathbb{Z} : n < 0\}$, $\{-n : n \in \mathbb{N}\}$
   (b) $\{2n + 1 : n \in \mathbb{Z}, 0 \leq n \leq 3\}$, $\{x \in \mathbb{R} : (x - 1)(x - 3)(x - 5)(x - 7) = 0\}$
   (c) $\{\frac{n}{n+1} : n \in \mathbb{N}\}$, $\{\frac{n-1}{n} : n \in \mathbb{Z}, n \geq 2\}$
   (d) $\{x \in \mathbb{Q} : 0 < x < 1\}$, $\{\frac{p}{q} : p, q \in \mathbb{N}, p < q\}$
   (e) $\{n \in \mathbb{N} : n \leq 10\}$, $\{n \in \mathbb{N} : n^2 < 101\}$

   Some ways of writing sets are of course more useful than others!

2. (a) Let A, B and C be sets. Which of the following is always true?
   i. $A \setminus (B \setminus C) = (A \setminus B) \cup C$
   ii. $A \setminus (B \cup C) = (A \setminus B) \setminus C$
   iii. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Solution

i. False
ii. True
iii. True

(b) For each statement which is true, give a proof. For each statement which is false, give an example of sets A, B and C such that the statement is false.

Solution

i. Let $A = \{0\}$, $B = \varnothing$ and $C = \{1\}$. Then, $A \setminus (B \setminus C) = A \setminus B = A = \{0\}$ and $(A \setminus B) \cup C = A \cup C = \{0, 1\}$.
ii. By definition,

$$\begin{aligned} A \setminus (B \cup C) &= A \cap (B \cup C)' \\ &= A \cap (B' \cap C') && \text{(de Morgan's laws)} \\ &= (A \cap B') \cap C' && \text{(associative law)} \\ &= (A \setminus B) \cap C' \\ &= (A \setminus B) \setminus C \end{aligned}$$

iii. By definition,

$$\begin{aligned} A \setminus (B \cap C) &= A \cap (B \cap C)' \\ &= A \cap (B' \cup C') && \text{(de Morgan's laws)} \\ &= (A \cap B') \cup (A \cap C') && \text{(distributive law)} \\ &= (A \setminus B) \cup (A \setminus C) \end{aligned}$$