Proof

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Week 4

Class Content

What types of proof are there?

- Direct proof
- Proof by contradiction
- Proof by induction
- Proof by exhaustion
- Counterexample

Question 1

Prove that for $n \in \mathbb{Z}$, if n is even if and only if n^3 is even.

Question 2

Prove that $\sqrt[3]{2}$ is irrational.

Question 3

Prove that $9^n - 1$ is divisible by 8 for every $n \in \mathbb{N}$.

Additional Questions

1. For $a, b, c \in \mathbb{Z}$, prove that if a divides b and b divides c, then a divides c.

2. Prove that no square number ends in a 7.

3. A sequence is defined recursively by $a_1 = 6$, $a_2 = 27$ and $a_{n+2} = 6a_{n+1} - 9a_n$ for $n \in \mathbb{N}$. Prove that $a_n = 3^n(n+1)$ for all $n \in \mathbb{N}$.

4. Prove by contradiction that if $x \in \mathbb{Q}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$, then $x + y \in \mathbb{R} \setminus \mathbb{Q}$.

5. Is it true that if $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, then $x + y \in \mathbb{Q}$? Provide a proof or a counterexample.

6. Is it true that if $x \in \mathbb{R} \backslash \mathbb{Q}$ and $y \in \mathbb{R} \backslash \mathbb{Q}$, then $x + y \in \mathbb{R} \backslash \mathbb{Q}$? Provide a proof or a counterexample.

7. All cows are the same colour.

   Proof. Let $P(n)$ be the statement that any n cows are all the same colour for $n \in \mathbb{N}$. Since one cow is always the same colour as itself, $P(1)$ is true.

   Suppose that $P(k)$ is true for some $k \in \mathbb{N}$, so any k cows are all the same colour. Let C be a set of $k + 1$ cows. If one cow c_1 is removed from C, then $C \setminus \{c_1\}$ is a set of k cows, so by the inductive hypothesis all cows in $C \setminus \{c_1\}$ are the same colour. Similarly, if a different cow c_2 is removed from C, then $C \setminus \{c_2\}$ is a set of k cows, so by the inductive hypothesis all cows in $C \setminus \{c_2\}$ are the same colour.

   Since $C \setminus \{c_1, c_2\} \subseteq C \setminus \{c_1\}$ and $C \setminus \{c_1, c_2\} \subseteq C \setminus \{c_2\}$, the cows in $C \setminus \{c_1, c_2\}$ are the same colour as the cows in $C \setminus \{c_1\}$ and the cows in $C \setminus \{c_2\}$. This implies that the cows in $C \setminus \{c_1\}$ and $C \setminus \{c_2\}$ are all the same colour.

   Since $C = (C \setminus \{c_1\}) \cup (C \setminus \{c_2\})$, this implies that all cows in C are the same colour, so $P(k+1)$ is true.

   Hence, by induction, all cows are the same colour. $\square$

   This result clearly is not true, so what is the mistake in the proof?