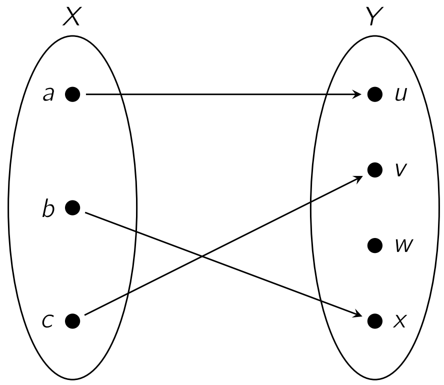
Week 5

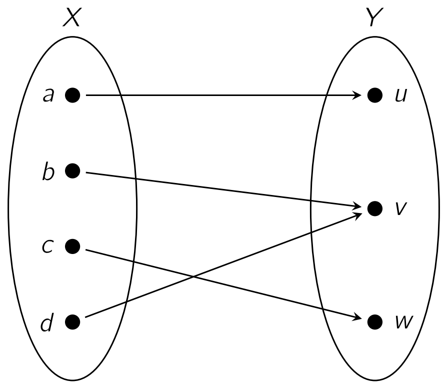
Class Content

Definition. *A function $f : X \to Y$ is* **injective** *if for every $y \in Y$, there exists at most one $x \in X$ such that $y = f(x)$.*

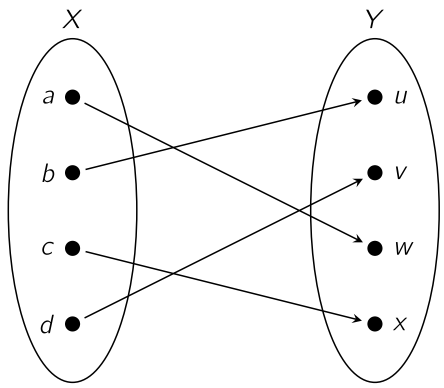
Alternatively, f is **injective** *if for every $w, x \in X$, $f(w) = f(x)$ implies that $w = x$.*



Definition. *A function $f : X \to Y$ is* **surjective** *if for every $y \in Y$, there exists at least one $x \in X$ such that $y = f(x)$.*



Definition. *A function $f : X \to Y$ is* **bijective** *if it is both injective and surjective, so for every $y \in Y$, there exists exactly one $x \in X$ such that $y = f(x)$.*

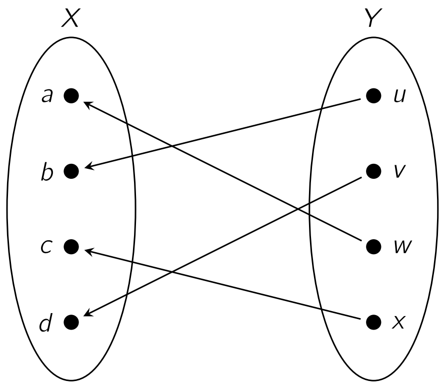


Definition. *A function $f : X \to Y$ is* **invertible** *if there is a function $f^{-1} : Y \to X$ such that $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$.*

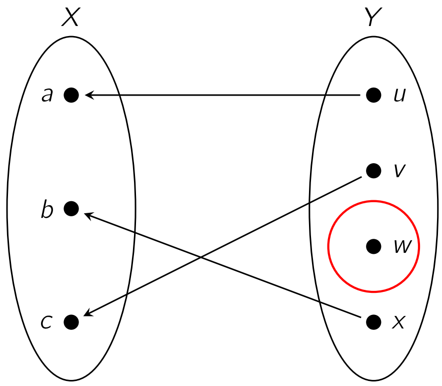
Theorem. *A function $f : X \to Y$ is bijective if and only if f is invertible.*

Consider the diagrams from above:

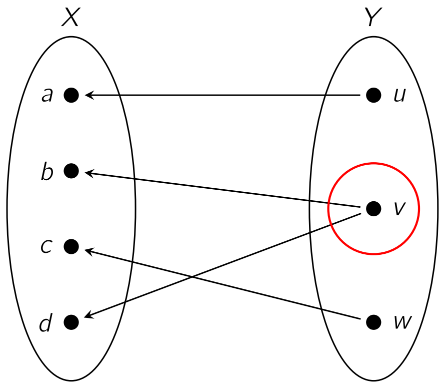
- Bijection:



- Injection:



- Surjection:



Question 1

Which of the following functions are injective, surjective or bijective?

1. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$
2. $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ where $f(x) = x^2$
3. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ where $f(x) = \frac{1}{x}$
4. $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ where $f(x) = \frac{1}{x}$

5. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ where $f(x) = \frac{1}{x}$
6. $f : \mathbb{N} \to \mathbb{Z}$ where
$$f(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ -\frac{x+1}{2} & x \text{ is odd} \end{cases}$$
7. $f : \mathbb{N} \to \mathbb{Z}$ where
$$f(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ -\frac{x-1}{2} & x \text{ is odd} \end{cases}$$
8. $f : \mathbb{N} \to \mathbb{N}$ where
$$f(x) = \begin{cases} 1 & x = 1 \\ x - 1 & x > 1 \end{cases}$$

Question 2

Prove that if a function $f : X \to Y$ is invertible, then it is bijective.

Additional Questions

1. Is it true that if a function $f : X \to Y$ is injective, then $|X| = |Y|$? Provide a proof or a counterexample.
2. Prove that for $f : X \to Y$ and $g : Y \to Z$, if f and g are both injective, then $g \circ f$ is injective.
3. Show that the converse (the converse is if $g \circ f$ is injective, then f and g are both injective) is not true with a counterexample.
4. Prove that for $f : X \to Y$ and $g : Y \to Z$, if $g \circ f$ is injective, then f is injective.
5. Prove that for $f : X \to Y$ and $g : Y \to Z$, if $g \circ f$ is surjective, then g is surjective.
6. We say a function $f : X \to Y$ has a left inverse if there is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ (this is similar to the definition of an inverse function, but only for composition on the left of f). Show that if f has a left inverse, then f is injective.
7. We say a function $f : X \to Y$ has a right inverse if there is a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ (this is similar to the definition of an inverse function, but only for composition on the right of f). Show that if f has a right inverse, then f is surjective.