# Functions

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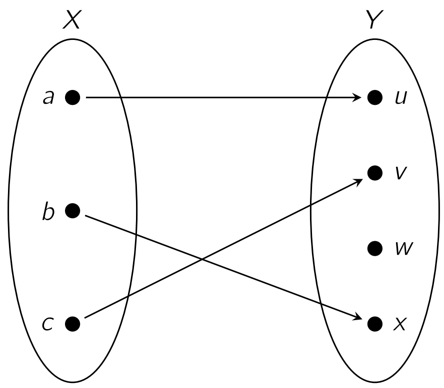
Week 5

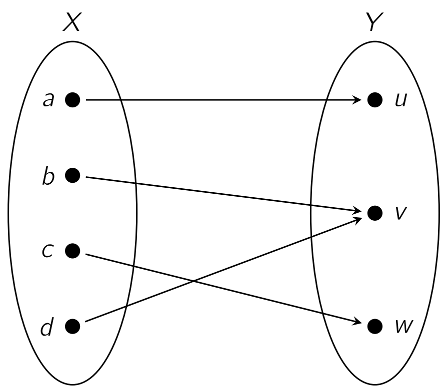
## Class Content

**Definition.** *A function $f : X \to Y$ is* **injective** *if for every $y \in Y$, there exists at most one $x \in X$ such that $y = f(x)$.*

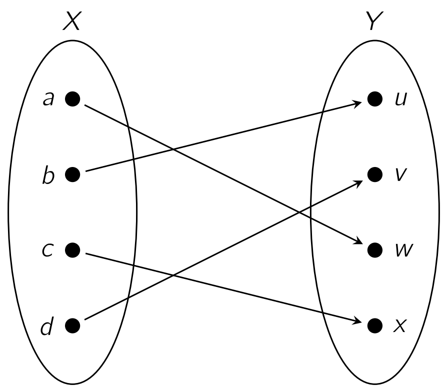
*Alternatively, $f$ is* **injective** *if for every $w, x \in X$, $f(w) = f(x)$ implies that $w = x$.*




**Definition.** *A function $f : X \to Y$ is* **surjective** *if for every $y \in Y$, there exists at least one $x \in X$ such that $y = f(x)$.*




**Definition.** *A function $f : X \to Y$ is* **bijective** *if it is both injective and surjective, so for every $y \in Y$, there exists exactly one $x \in X$ such that $y = f(x)$.*

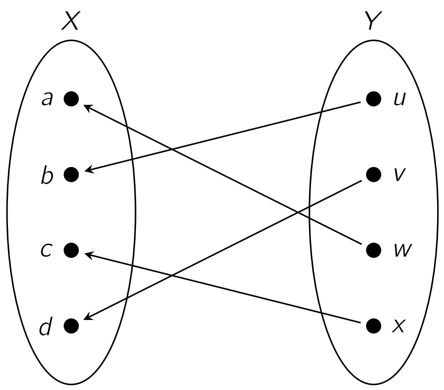



**Definition.** *A function $f : X \to Y$ is* **invertible** *if there is a function $f^{-1} : Y \to X$ such that $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$.*

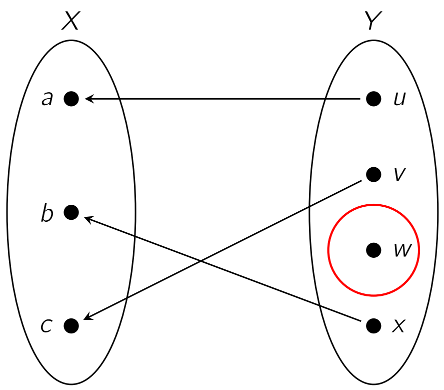
**Theorem.** *A function $f : X \to Y$ is bijective if and only if $f$ is invertible.*

Consider the diagrams from above:

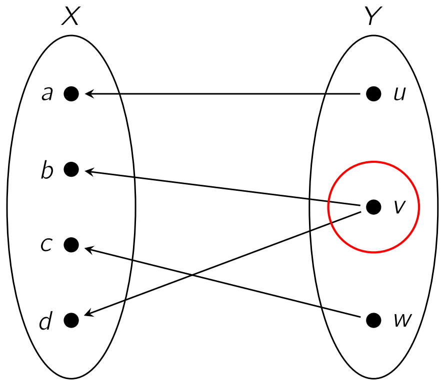
- Bijection:




- Injection:




- Surjection:




## Question 1

Which of the following functions are injective, surjective or bijective?

1. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$
2. $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ where $f(x) = x^2$
3. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ where $f(x) = \frac{1}{x}$
4. $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ where $f(x) = \frac{1}{x}$
5. $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ where $f(x) = \frac{1}{x}$
6. $f : \mathbb{N} \to \mathbb{Z}$ where
$$f(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ -\frac{x+1}{2} & x \text{ is odd} \end{cases}$$
7. $f : \mathbb{N} \to \mathbb{Z}$ where
$$f(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ -\frac{x-1}{2} & x \text{ is odd} \end{cases}$$
8. $f : \mathbb{N} \to \mathbb{N}$ where
$$f(x) = \begin{cases} 1 & x = 1 \\ x - 1 & x > 1 \end{cases}$$

### Solution

1. Not injective as $f(-1) = f(1)$, not surjective as there is no $x \in \mathbb{R}$ such that $f(x) = -1$.
2. Surjective as $f(\sqrt{y}) = y$ for $y \in \mathbb{R}_{\geq 0}$, not injective as $f(-1) = f(1)$.
3. Injective as
$$\begin{aligned} f(w) = f(x) &\implies \frac{1}{w} = \frac{1}{x} \\ &\implies w = x \end{aligned}$$
not surjective as there is no $x \in \mathbb{R} \setminus \{0\}$ such that $f(x) = 0$.
4. Bijective as $f$ is self-inverse ($f \circ f = \mathrm{id}_{\mathbb{R}_{>0}}$).
5. Bijective as $f$ is self-inverse ($f \circ f = \mathrm{id}_{\mathbb{R} \setminus \{0\}}$).

6. Injective as $f(x) > 0$ for even $x$ and $f(x) < 0$ for odd $x$ and

$$\begin{aligned} f(2w) = f(2x) &\implies w = x \\ &\implies 2w = 2x \end{aligned}$$

and

$$\begin{aligned} f(2w-1) = f(2x-1) &\implies -w = -x \\ &\implies 2w-1 = 2x-1 \end{aligned}$$

not surjective as there is no $x \in \mathbb{N}$ such that $f(x) = 0$.

7. Bijective as there is an inverse $f^{-1} : \mathbb{Z} \to \mathbb{N}$ where

$$f^{-1}(x) = \begin{cases} 2x & x > 0 \\ 1 - 2x & x \leq 0 \end{cases}$$

8. Not injective as $f(1) = f(2)$, surjective as $f(y+1) = y$ for $y \in \mathbb{N}$.

### Question 2

Prove that if a function $f : X \to Y$ is invertible, then it is bijective.

**Solution** Suppose that $f$ is invertible, so there is a function $f^{-1} : Y \to X$ where $(f \circ f^{-1})(y) = y$ for all $y \in Y$ and $(f^{-1} \circ f)(x) = x$ for all $x \in X$.

To show that $f$ is injective, suppose that $f(w) = f(x)$ for $w, x \in X$. By applying $f^{-1}$ to both sides, $f^{-1}(f(w)) = f^{-1}(f(x))$, so $(f^{-1} \circ f)(w) = (f^{-1} \circ f)(x)$. This implies that $w = x$, so $f$ is injective.

To show that $f$ is surjective, let $y \in Y$, so $f^{-1}(y) \in X$. By definition, $f(f^{-1}(y)) = (f \circ f^{-1})(y) = y$, so there exists $x \in X$ such that $f(x) = y$, so $f$ is surjective.

Hence, $f$ is bijective.

## Additional Questions

1. Is it true that if a function $f : X \to Y$ is injective, then $|X| = |Y|$? Provide a proof or a counterexample.

   **Solution** Let $X = \{0, 1\}$ and $Y = \{0, 1, 2\}$, so $|X| < |Y|$. Let $f : X \to Y$ be the function where $f(0) = 0$ and $f(1) = 1$. Since $f(w) \neq f(x)$ whenever $w \neq x$ for $w, x \in X$, $f$ is injective.

2. For $f : X \to Y$ and $g : Y \to Z$, if $f$ and $g$ are both injective, then $g \circ f$ is injective. Show that the converse is not true with a counterexample.

   **Solution** Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ where $f(x) = x$ and $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x^2$, so $g$ is not injective since $g(-1) = g(1)$.

   Then, $g \circ f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ where $(g \circ f)(x) = x^2$. If $(g \circ f)(w) = (g \circ f)(x)$, then $w^2 = x^2$ and since $w, x \in \mathbb{R}_{\geq 0}$, this implies that $w = x$, so $g \circ f$ is injective.

   Hence $g \circ f$ is injective but $g$ is not injective, so the converse cannot hold.

3. Prove that for $f : X \to Y$ and $g : Y \to Z$, if $g \circ f$ is injective, then $f$ is injective.

**Solution** Since $g \circ f$ is injective, for every $w, x \in X$, if $(g \circ f)(w) = (g \circ f)(x)$, then $w = x$.

For every $w, x \in X$, if $f(w) = f(x)$, then $g(f(w)) = g(f(x))$, so $(g \circ f)(w) = (g \circ f)(x)$. Using the injectivity of $g \circ f$, this implies that $w = x$. Hence, $f$ is injective.

4. For $f : X \to Y$ and $g : Y \to Z$, if $f$ and $g$ are both surjective, then $g \circ f$ is surjective. Show that the converse is not true with a counterexample.

**Solution** Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ where $f(x) = \sqrt{x}$ and $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ where $g(x) = |x|$, so $f$ is not surjective since there is no $x \in \mathbb{R}_{\geq 0}$ with $f(x) = -1$.

Then, $g \circ f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ where $(g \circ f)(x) = |\sqrt{x}| = \sqrt{x}$. For all $y \in \mathbb{R}_{\geq 0}$, $y^2 \in \mathbb{R}_{\geq 0}$ and $(g \circ f)(y^2) = y$, so $g \circ f$ is surjective.

Hence, $g \circ f$ is surjective but $f$ is not surjective, so the converse cannot hold.

5. Prove that for $f : X \to Y$ and $g : Y \to Z$, if $g \circ f$ is surjective, then $g$ is surjective.

**Solution** Since $g \circ f$ is surjective, for every $z \in Z$, there exists $x \in X$ such that $z = (g \circ f)(x)$. Since $f(x) \in Y$, this implies that for every $z \in Z$, there exists $y \in Y$ such that $z = g(y)$, by taking $y = f(x)$ where $z = (g \circ f)(x)$. Hence, $g$ is surjective.

6. We say a function $f : X \to Y$ has a left inverse if there is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ (this is similar to the definition of an inverse function, but only for composition on the left of $f$). Show that if $f$ has a left inverse, then $f$ is injective.

**Solution** Suppose that $f$ has a left inverse $g : Y \to X$. By definition, $(g \circ f)(x) = x$ for every $x \in X$. Suppose that $f(w) = f(x)$ for $w, x \in X$. This implies that $g(f(w)) = g(f(x))$, so $(g \circ f)(w) = (g \circ f)(x)$ and hence $w = x$. This implies that $f$ is injective.

7. We say a function $f : X \to Y$ has a right inverse if there is a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ (this is similar to the definition of an inverse function, but only for composition on the right of $f$). Show that if $f$ has a right inverse, then $f$ is surjective.

**Solution** Suppose that $f$ has a right inverse $g : Y \to X$. By definition, $(f \circ g)(y) = y$ for every $y \in Y$. Since $g(y) \in X$, this implies that for every $y \in Y$, there exists $x = g(y)$ such that $y = f(x)$. This implies that $f$ is surjective.