Sequences and Series

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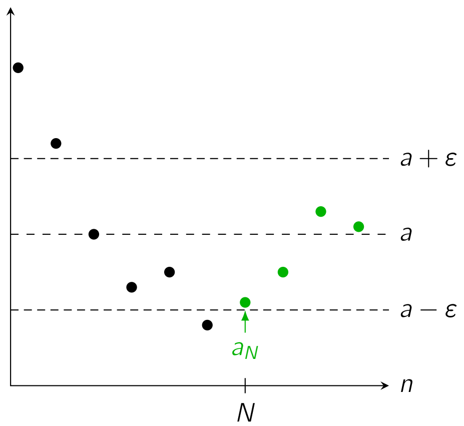
Week 6

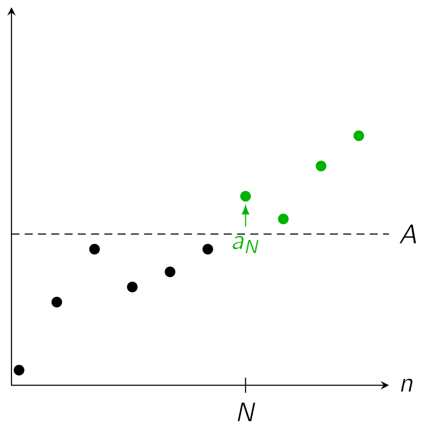
Class Content

Definition. *For a sequence (a_n) and $a \in \mathbb{R}$, $(a_n) \to a$ if for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for every $n > N$.*



Definition. *For a sequence (a_n), $(a_n) \to \infty$ if for all $A > 0$, there is some $N \in \mathbb{N}$ such that $a_n > A$ for every $n > N$.*

Similarly, $(a_n) \to -\infty$ if for all $A < 0$, there is some $N \in \mathbb{N}$ such that $a_n < A$ for every $n > N$.



Question 1

Use the definition of convergence to show that the sequence $a_n = \frac{n}{n+1}$ converges to 1 as $n \to \infty$.

Theorem. *If $(a_n) \to a$ and $(b_n) \to b$, then*

- *$(a_n + b_n) \to a + b$ (sum rule)*

- $(a_n b_n) \to ab$ *(product rule)*
- $(\frac{a_n}{b_n}) \to \frac{a}{b}$ *if* $b \neq 0$ *(quotient rule)*

Theorem (Sandwich Rule)**.** *For sequences* (a_n), (b_n) *and* (c_n), *if* $(a_n) \to L$ *and* $(c_n) \to L$ *and there is some* $N \in \mathbb{N}$ *such that* $a_n \leq b_n \leq c_n$ *for all* $n > N$, *then* $(b_n) \to L$.

Question 2

Find the limits of the following sequences:

1. $a_n = \frac{2n^2+3n}{n^3+n^2}$
2. $a_n = \frac{3n^2+n\cos n}{2n(n-3)}$

Definition. *For a sequence* (a_k), *the corresponding* **series** *is the sum*

$$\sum_{k=1}^{\infty} a_k$$

Definition. *For a sequence* (a_k) *and* $S \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} a_k = S$$

if the sequence of partial sums $(S_n) \to S$ *where*

$$S_n = \sum_{k=1}^{n} a_k$$

Theorem (Sum Rule)**.** *For series* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$, *if* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *both converge, then* $\sum_{k=1}^{\infty}(a_k + b_k)$ *converges.*

Theorem (Null Sequence Test)**.** *The series* $\sum_{k=1}^{\infty} a_k$ *only converges if* $(a_k) \to 0$.

Theorem (Comparison Test)**.** *For series* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$, *if*

- $a_k, b_k \geq 0$ *for all* $k \in \mathbb{N}$
- $a_k \leq Mb_k$ *for all* $k \in \mathbb{N}$ *and* $M > 0$
- $\sum_{k=1}^{\infty} b_k$ *converges*

then $\sum_{k=1}^{\infty} a_k$ *converges. Similarly, if*

- $a_k, b_k \geq 0$ *for all* $k \in \mathbb{N}$
- $a_k \geq Mb_k$ *for all* $k \in \mathbb{N}$ *and* $M > 0$
- $\sum_{k=1}^{\infty} b_k$ *diverges*

then $\sum_{k=1}^{\infty} a_k$ *diverges.*

Theorem (Ratio Test)**.** *For a series* $\sum_{k=1}^{\infty} a_k$, *if* $(|\frac{a_{k+1}}{a_k}|) \to L$, *then*

- *if* $L < 1$, $\sum_{k=1}^{\infty} a_k$ *converges*
- *if* $L > 1$, $\sum_{k=1}^{\infty} a_k$ *diverges*

Theorem (Alternating Series Test)**.** *For a sequence* (a_k), *if*

- $a_k > 0$ *for all* $k \in \mathbb{N}$
- (a_k) *is decreasing*
- $(a_k) \to 0$

then $\sum_{k=1}^{\infty}(-1)^k a_k$ *converges.*

Question 3

Which of the following series are convergent?

1. $\sum_{n=1}^{\infty} \frac{n^2+n^3}{n^5}$
2. $\sum_{n=1}^{\infty} (-1)^n$
3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
4. $\sum_{n=1}^{\infty} \frac{1}{n!}$
5. $\sum_{n=1}^{\infty} \frac{3^n}{n}$
6. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+2}{n^{\frac{3}{2}}+1}$

Additional Questions

1. Use the definition of convergence to show that $(\frac{1}{\sqrt{n}}) \to 0$ as $n \to \infty$.

2. Use the triangle inequality and the definition of convergence to show that if $(a_n) \to a$ and $(a_n) \to b$, then $a = b$ (i.e. that limits of sequences are unique).

3. Decide whether the series
$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^3+1}$$
converges or diverges.

4. (a) Find a series $\sum_{k=1}^{\infty} a_k$ that converges where $(|\frac{a_{k+1}}{a_k}|) \to 1$.
   (b) Find a series $\sum_{k=1}^{\infty} a_k$ that diverges where $(|\frac{a_{k+1}}{a_k}|) \to 1$.

   This shows that the ratio test is inconclusive when $L = 1$.