Sequences and Series

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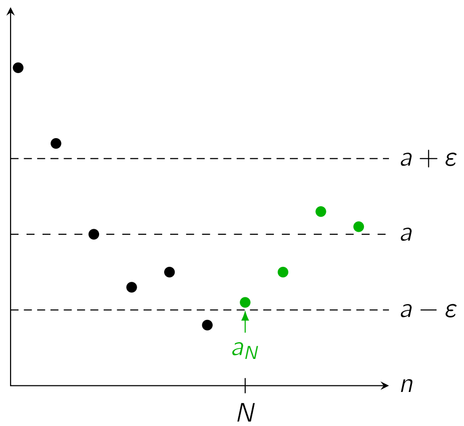
Week 6

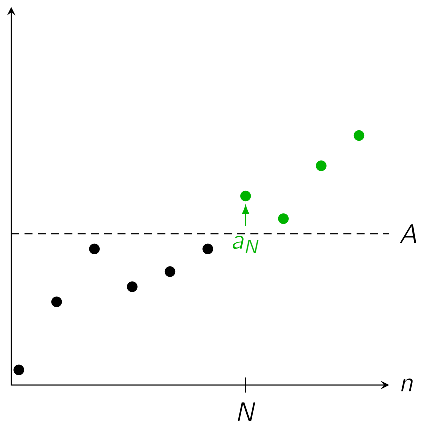
Class Content

Definition. *For a sequence (a_n) and $a \in \mathbb{R}$, $(a_n) \to a$ if for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for every $n > N$.*



Definition. *For a sequence (a_n), $(a_n) \to \infty$ if for all $A > 0$, there is some $N \in \mathbb{N}$ such that $a_n > A$ for every $n > N$.*

Similarly, $(a_n) \to -\infty$ if for all $A < 0$, there is some $N \in \mathbb{N}$ such that $a_n < A$ for every $n > N$.



Question 1

Use the definition of convergence to show that the sequence $a_n = \frac{n}{n+1}$ converges to 1 as $n \to \infty$.

Solution To prove that $(a_n) \to 1$, we need to show that for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - 1| < \varepsilon$ for every $n > N$.

Let $\varepsilon > 0$. By definition,

$$\begin{aligned} a_n - 1 &= \frac{n}{n+1} - 1 \\ &= \frac{n-n-1}{n+1} \\ &= -\frac{1}{n+1} \\ \implies |a_n - 1| &= \frac{1}{n+1} \\ &< \frac{1}{n} \end{aligned}$$

Let $N \in \mathbb{N}$ where $N \geq \frac{1}{\varepsilon}$. Then, for all $n > N$

$$\begin{aligned} |a_n - 1| &< \frac{1}{n} \\ &< \frac{1}{N} && \text{(since } n > N\text{)} \\ &\leq \varepsilon && \text{(since } N \geq \frac{1}{\varepsilon}\text{)} \end{aligned}$$

Hence, $(a_n) \to 1$.

Theorem. *If $(a_n) \to a$ and $(b_n) \to b$, then*

- $(a_n + b_n) \to a + b$ *(sum rule)*
- $(a_n b_n) \to ab$ *(product rule)*
- $(\frac{a_n}{b_n}) \to \frac{a}{b}$ *if $b \neq 0$ (quotient rule)*

Theorem (Sandwich Rule)**.** *For sequences (a_n), (b_n) and (c_n), if $(a_n) \to L$ and $(c_n) \to L$ and there is some $N \in \mathbb{N}$ such that $a_n \leq b_n \leq c_n$ for all $n > N$, then $(b_n) \to L$.*

Question 2

Find the limits of the following sequences:

1. $a_n = \frac{2n^2+3n}{n^3+n^2}$
2. $a_n = \frac{3n^2+n\cos n}{2n(n-3)}$

Solution

1. By definition,

$$\begin{aligned} a_n &= \frac{2n^2+3n}{n^3+n^2} \\ &= \frac{\frac{2}{n}+\frac{3}{n^2}}{1+\frac{1}{n}} \end{aligned}$$

   Since $(\frac{1}{n}) \to 0$, $(\frac{2}{n}) \to 0$ and $(\frac{3}{n^2}) \to 0$ by the product rule. This implies that $(\frac{2}{n} + \frac{3}{n^2}) \to 0$ and $(1 + \frac{1}{n}) \to 1$ by the sum rule. Hence, by the quotient rule, $(a_n) \to 0$.

2. By definition,

$$\begin{aligned} a_n &= \frac{3n^2+n\cos n}{2n(n-3)} \\ &= \frac{3n^2+n\cos n}{2n^2-6n} \\ &= \frac{3+\frac{\cos n}{n}}{2-\frac{6}{n}} \end{aligned}$$

Since $(\frac{1}{n}) \to 0$, $(\frac{6}{n}) \to 0$ by the product rule.

Since $-1 \leq \cos n \leq 1$, $\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Since $(\frac{1}{n}) \to 0$ and $(\frac{-1}{n}) \to 0$, this implies that $(\frac{\cos n}{n}) \to 0$ by the sandwich rule.

This implies that $(3 + \frac{\cos n}{n}) \to 3$ and $(2 - \frac{6}{n^2}) \to 2$ by the sum rule. Hence, by the quotient rule, $(a_n) \to \frac{3}{2}$.

Definition. *For a sequence (a_k), the corresponding* **series** *is the sum*

$$\sum_{k=1}^{\infty} a_k$$

Definition. *For a sequence (a_k) and $S \in \mathbb{R}$,*

$$\sum_{k=1}^{\infty} a_k = S$$

if the sequence of partial sums $(S_n) \to S$ where

$$S_n = \sum_{k=1}^{n} a_k$$

Theorem (Sum Rule)**.** *For series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, then $\sum_{k=1}^{\infty}(a_k + b_k)$ converges.*

Theorem (Null Sequence Test)**.** *The series $\sum_{k=1}^{\infty} a_k$ only converges if $(a_k) \to 0$.*

Theorem (Comparison Test)**.** *For series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, if*

- *$a_k, b_k \geq 0$ for all $k \in \mathbb{N}$*
- *$a_k \leq Mb_k$ for all $k \in \mathbb{N}$ and $M > 0$*
- *$\sum_{k=1}^{\infty} b_k$ converges*

then $\sum_{k=1}^{\infty} a_k$ converges. Similarly, if

- *$a_k, b_k \geq 0$ for all $k \in \mathbb{N}$*
- *$a_k \geq Mb_k$ for all $k \in \mathbb{N}$ and $M > 0$*
- *$\sum_{k=1}^{\infty} b_k$ diverges*

then $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem (Ratio Test)**.** *For a series $\sum_{k=1}^{\infty} a_k$, if $(|\frac{a_{k+1}}{a_k}|) \to L$, then*

- *if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges*
- *if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges*

Theorem (Alternating Series Test)**.** *For a sequence (a_k), if*

- *$a_k > 0$ for all $k \in \mathbb{N}$*
- *(a_k) is decreasing*
- *$(a_k) \to 0$*

then $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

Question 3

Which of the following series are convergent?

1. $\sum_{n=1}^{\infty} \frac{n^2+n^3}{n^5}$
2. $\sum_{n=1}^{\infty}(-1)^n$
3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
4. $\sum_{n=1}^{\infty} \frac{1}{n!}$
5. $\sum_{n=1}^{\infty} \frac{3^n}{n}$
6. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+2}{n^{\frac{3}{2}}+1}$

Solution

1. Convergent, using the sum rule. Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^3}$, so

$$a_n + b_n = \frac{1}{n^2} + \frac{1}{n^3} = \frac{n^2+n^3}{n^5}$$

   Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, the series converges.

2. Divergent, using the null sequence test. Let $a_n = (-1)^n$. Since (a_n) is not convergent, the series diverges.

3. Convergent, using the alternating series test. Let $a_n = \frac{1}{n^2}$, so (a_n) is decreasing and $(a_n) \to 0$, and so the series converges.

4. Convergent, using the ratio test. Let $a_n = \frac{1}{n!}$, so

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

   Since $(\frac{1}{n+1}) \to 0$, the series converges.

5. Divergent, using the ratio test. Let $a_n = \frac{3^n}{n}$, so

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{3^{n+1}n}{3^n(n+1)} \\ &= \frac{3n}{n+1} \\ &= \frac{3}{1+\frac{1}{n}}\end{aligned}$$

   Using the sum rule and the quotient rule, $(\frac{3}{1+\frac{1}{n}}) \to 3$, so the series diverges.

6. Divergent, using the comparison test. Let $a_n = \frac{\sqrt{n}+2}{n^{\frac{3}{2}}+1}$, so

$$\begin{aligned}\frac{\sqrt{n}+2}{n^{\frac{3}{2}}+1} &= \frac{1+\frac{2}{\sqrt{n}}}{n+\frac{1}{\sqrt{n}}} \\ &> \frac{1+\frac{2}{\sqrt{n}}}{2n} \\ &> \frac{1}{2n}\end{aligned}$$

   Since $a_n > \frac{1}{2}(\frac{1}{n})$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series diverges.

Additional Questions

1. Use the definition of convergence to show that $(\frac{1}{\sqrt{n}}) \to 0$ as $n \to \infty$.

   Solution By definition, $(\frac{1}{\sqrt{n}}) \to 0$ if for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, $|a_n| < \varepsilon$.

   Let $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that $N \geq \frac{1}{\varepsilon^2}$. Then, for all $n > N$,

$$\begin{aligned} |a_n| &= \frac{1}{\sqrt{n}} \\ &< \frac{1}{\sqrt{N}} && \text{(since } n > N\text{)} \\ &\leq \varepsilon && \text{(since } N \geq \frac{1}{\varepsilon^2}\text{)} \end{aligned}$$

   so $(\frac{1}{\sqrt{n}}) \to 0$.

2. Use the triangle inequality and the definition of convergence to show that if $(a_n) \to a$ and $(a_n) \to b$, then $a = b$ (i.e. that limits of sequences are unique).

   Solution If $(a_n) \to a$, then by definition, for all $\varepsilon > 0$, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $|a_n - a| < \varepsilon$.

   If $(a_n) \to b$, then by definition, for all $\varepsilon > 0$, there is $N_2 \in \mathbb{N}$ such that for all $n > N_2$, $|a_n - b| < \varepsilon$.

   By the triangle inequality,

$$\begin{aligned} |a - b| &= |a - a_n + a_n - b| \\ &\leq |a - a_n| + |a_n - b| \\ &= |a_n - a| + |a_n - b| \end{aligned}$$

   This implies that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ where $N = \max(N_1, N_2)$ such that for all $n > N$,

$$\begin{aligned} |a - b| &\leq |a_n - a| + |a_n - b| \\ &< 2\varepsilon \end{aligned}$$

   Since this holds for every $\varepsilon > 0$, this implies that $|a - b| = 0$ and so $a = b$.

3. Decide whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^3 + 1}$$

   converges or diverges.

   Solution Let $a_n = \frac{n^2}{n^3+1}$, so $a_n > 0$ for all $n \in \mathbb{N}$. Since

$$\begin{aligned} a_n &= \frac{n^2}{n^3 + 1} \\ &= \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} \end{aligned}$$

   $(a_n) \to 0$ by the sum rule, product rule and quotient rule.

By definition, for all $n \in \mathbb{N}$,

$$\begin{aligned}
a_{n+1} - a_n &= \frac{(n+1)^2}{(n+1)^3+1} - \frac{n^2}{n^3+1} \\
&= \frac{n^2+2n+1}{n^3+3n^2+3n+2} - \frac{n^2}{n^3+1} \\
&= \frac{(n^2+2n+1)(n^3+1) - n^2(n^3+3n^2+3n+2)}{(n^3+1)(n^3+3n^2+3n+2)} \\
&= \frac{(n^5+2n^4+n^3+n^2+2n+1) - (n^5+3n^4+3n^3+2n^2)}{(n^3+1)(n^3+3n^2+3n+2)} \\
&= \frac{-n^4-2n^3-n^2+2n+1}{(n^3+1)(n^3+3n^2+3n+2)} \\
&< \frac{-2-n^2+2n+1}{(n^3+1)(n^3+3n^2+3n+2)} && \text{(since } n^4+2n^3>2) \\
&= \frac{-(n^2-2n+1)}{(n^3+1)(n^3+3n^2+3n+2)} \\
&= \frac{-(n-1)^2}{(n^3+1)(n^3+3n^2+3n+2)} \\
&\leq 0 && \text{(since } (n-1)^2 \geq 0)
\end{aligned}$$

so $a_{n+1} - a_n < 0$ and hence $a_n > a_{n+1}$. Since $a_n > 0$ for all $n \in \mathbb{N}$, $(a_n) \to 0$ and (a_n) is decreasing, by the alternating series test, the series converges.

4. (a) Find a series $\sum_{k=1}^{\infty} a_k$ that converges where $(|\frac{a_{k+1}}{a_k}|) \to 1$.
   (b) Find a series $\sum_{k=1}^{\infty} a_k$ that diverges where $(|\frac{a_{k+1}}{a_k}|) \to 1$.

   This shows that the ratio test is inconclusive when $L = 1$.

   Solution

   (a) Let $a_n = \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} a_n$ converges. Since

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \\
&= \frac{n^2}{(n+1)^2} \\
&= \frac{n^2}{n^2+2n+1} \\
&= \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}}
\end{aligned}$$

   $\left(\left|\frac{a_{n+1}}{a_n}\right|\right) \to 1$ by the sum rule, product rule and quotient rule.

   (b) Let $a_n = \frac{1}{n}$, so $\sum_{n=1}^{\infty} a_n$ diverges. Since

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\frac{1}{n+1}}{\frac{1}{n}} \\
&= \frac{n}{n+1} \\
&= \frac{1}{1+\frac{1}{n}}
\end{aligned}$$

   $\left(\left|\frac{a_{n+1}}{a_n}\right|\right) \to 1$ by the sum rule and quotient rule.