Limits and Continuity

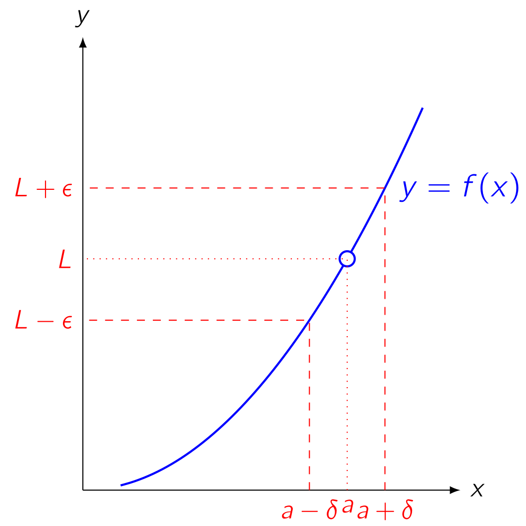
Michael Cavaliere

Week 7

Class Content

Definition. *For a real-valued function f and $a, L \in \mathbb{R}$, the* **limit** *of $f(x)$ is L as $x \to a$ if for all $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.*

This is denoted by $f(x) \to L$ as $x \to a$, or as $\lim_{x \to a} f(x) = L$.



Theorem. *If f and g are real-valued functions with $f(x) \to L$ and $g(x) \to M$ as $x \to a$, then*

- $f(x) + g(x) \to L + M$ *(sum rule)*
- $f(x)g(x) \to LM$ *(product rule)*
- $\frac{f(x)}{g(x)} \to \frac{L}{M}$ *if $M \neq 0$ (quotient rule)*

Theorem (Sandwich Rule)**.** *For real-valued functions f, g and h, if $f(x) \to L$ and $h(x) \to L$ as $x \to a$ and $f(x) \leq g(x) \leq h(x)$, then $g(x) \to L$ as $x \to a$.*

Question 1

Find the following limits:

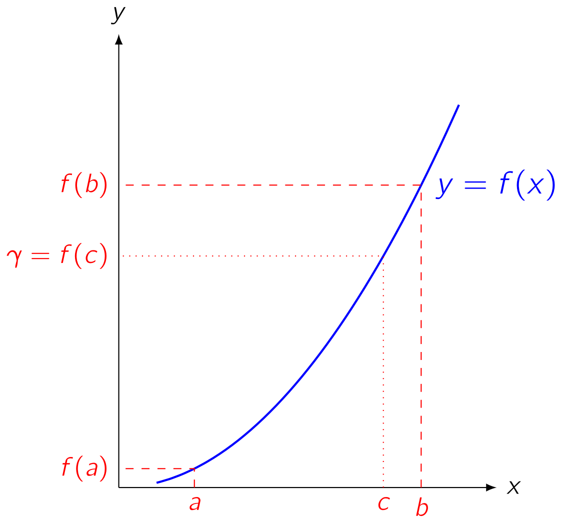
1. $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}$
2. $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$
3. $\lim_{x \to 0} x \sin(\frac{1}{x})$

Question 2

Use the substitution $x = \frac{1}{t}$ and consider $t \to 0$ to find $\lim_{x\to\infty} \frac{5x^3+2x^2-7}{x^4+3x}$.

Definition. *For a real-valued function f and $a \in \mathbb{R}$, $f(x)$ is* **continuous** *at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.*
The function f is **continuous** *on an interval if it is continuous at every point in the interval.*

Theorem (Intermediate Value Theorem)**.** *For a function $f : [a, b] \to \mathbb{R}$, if f is continuous on the interval $[a, b]$ with $f(a) = \alpha$ and $f(b) = \beta$, then for any $\gamma \in (\alpha, \beta)$, there is $c \in (a, b)$ such that $\gamma = f(c)$.*



Question 3

Show that the polynomial $f(x) = 3x^3 + x^2 - 6x + 1$ has three real roots in the interval $[-2, 2]$.

Additional Questions

1. Calculate the following limits, clearly stating what rules you used:
   (a) $\lim_{x\to 8} \frac{2x^2-17x+8}{8-x}$
   (b) $\lim_{x\to 0} \frac{x}{3-\sqrt{x+9}}$
   (c) $\lim_{x\to\infty} x \sin(\frac{\pi}{x})$
   (d) $\lim_{x\to\infty} \frac{1+x}{x^x}$ (Hint: use the fact that $x^x \geq x^2$ for all $x \geq 2$.)

2. Use the following theorem to find the limits of
   (a) e^{4x+1} as $x \to -\frac{1}{2}$.
   (b) $2^{\sin(x)}$ as $x \to \frac{\pi}{2}$.

   Theorem. *If f and g are real-valued functions where $\lim_{x\to a} f(x) = L$ and g is continuous at $x = L$, then $\lim_{x\to a}(g \circ f)(x) = g(\lim_{x\to a} f(x)) = g(L)$.*

3. Use the definitions of convergence of sequences and continuity to show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $(a_n) \to a$, then $(f(a_n)) \to f(a)$.