Limits and Continuity

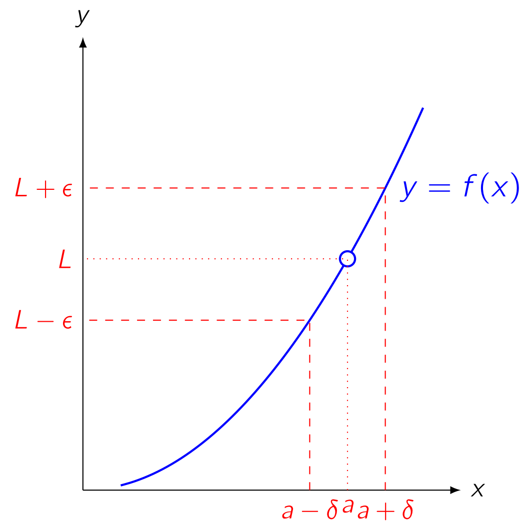
Michael Cavaliere

Week 7

Class Content

Definition. *For a real-valued function f and $a, L \in \mathbb{R}$, the* **limit** *of $f(x)$ is L as $x \to a$ if for all $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.*

This is denoted by $f(x) \to L$ as $x \to a$, or as $\lim_{x \to a} f(x) = L$.



Theorem. *If f and g are real-valued functions with $f(x) \to L$ and $g(x) \to M$ as $x \to a$, then*

- $f(x) + g(x) \to L + M$ *(sum rule)*
- $f(x)g(x) \to LM$ *(product rule)*
- $\frac{f(x)}{g(x)} \to \frac{L}{M}$ *if $M \neq 0$ (quotient rule)*

Theorem (Sandwich Rule)**.** *For real-valued functions f, g and h, if $f(x) \to L$ and $h(x) \to L$ as $x \to a$ and $f(x) \leq g(x) \leq h(x)$, then $g(x) \to L$ as $x \to a$.*

Question 1

Find the following limits:

1. $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}$
2. $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$
3. $\lim_{x \to 0} x \sin(\frac{1}{x})$

Solution

1. Since $x^2 - 3x + 2 = (x-1)(x-2)$,

$$\begin{aligned}\lim_{x\to 1} \frac{x^2 - 3x + 2}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x-2)}{x-1} \\ &= \lim_{x\to 1}(x-2) \\ &= (\lim_{x\to 1} x) - 2 \\ &= -1\end{aligned}$$

   by the sum rule.

2. Since $x^2 - 4 = (x+2)(x-2)$,

$$\begin{aligned}\lim_{x\to 2} \frac{x-2}{x^2-4} &= \lim_{x\to 2} \frac{x-2}{(x+2)(x-2)} \\ &= \lim_{x\to 2} \frac{1}{x+2} \\ &= \frac{1}{\lim_{x\to 2}(x+2)} \\ &= \frac{1}{(\lim_{x\to 2} x) + 2} \\ &= \frac{1}{4}\end{aligned}$$

   by the sum rule and the quotient rule.

3. Since $-1 \leq \sin(\frac{1}{x}) \leq 1$, $-x \leq x\sin(\frac{1}{x}) \leq x$ and so, since $\lim_{x\to 0} x = \lim_{x\to 0}(-x) = 0$, by the sandwich rule $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$.

Question 2

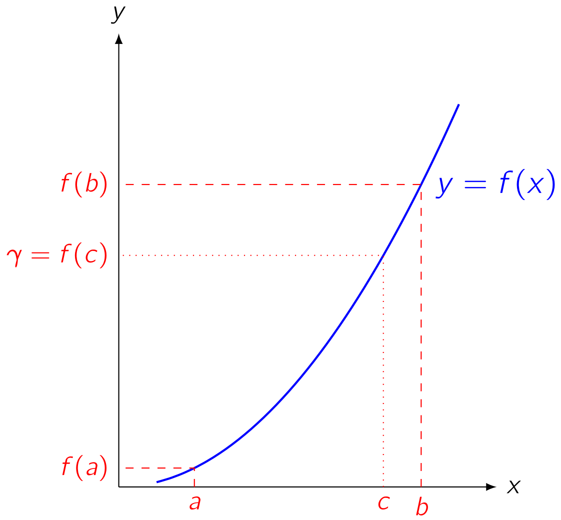
Use the substitution $x = \frac{1}{t}$ and consider $t \to 0$ to find $\lim_{x\to\infty} \frac{5x^3+2x^2-7}{x^4+3x}$.

Solution Let $x = \frac{1}{t}$, so

$$\begin{aligned}\lim_{x\to\infty} \frac{5x^3 + 2x^2 - 7}{x^4 + 3x} &= \lim_{t\to 0} \frac{\frac{5}{t^3} + \frac{2}{t^2} - 7}{\frac{1}{t^4} + \frac{3}{t}} \\ &= \lim_{t\to 0} \frac{\frac{5t+2t^2-7t^4}{t^4}}{\frac{1+3t^3}{t^4}} \\ &= \lim_{t\to 0} \frac{5t + 2t^2 - 7t^4}{1 + 3t^3} \\ &= \frac{\lim_{t\to 0}(5t + 2t^2 - 7t^4)}{\lim_{t\to 0}(1 + 3t^3)} \\ &= 0\end{aligned}$$

Definition. *For a real-valued function f and $a \in \mathbb{R}$, $f(x)$ is* **continuous** *at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.*
The function f is **continuous** *on an interval if it is continuous at every point in the interval.*

Theorem (Intermediate Value Theorem)**.** *For a function $f : [a,b] \to \mathbb{R}$, if f is continuous on the interval $[a,b]$ with $f(a) = \alpha$ and $f(b) = \beta$, then for any $\gamma \in (\alpha, \beta)$, there is $c \in (a,b)$ such that $\gamma = f(c)$.*



Question 3

Show that the polynomial $f(x) = 3x^3 + x^2 - 6x + 1$ has three real roots in the interval $[-2, 2]$.

Solution By definition,

$$\begin{aligned} f(-2) &= -7 \\ f(-1) &= 5 \\ f(0) &= 1 \\ f(1) &= -1 \\ f(2) &= 17 \end{aligned}$$

Since f is a polynomial, f is continuous on $[-2, -1]$ so by the intermediate value theorem, for every $\gamma \in (-7, 5)$, there is some $c \in (-2, -1)$ such that $\gamma = f(c)$. That implies that there is a root of f in $(-2, -1)$ by taking $\gamma = 0$.

By the same argument applied to the intervals $[0, 1]$ and $[1, 2]$, there are roots of f in the intervals $(0, 1)$ and $(1, 2)$, so f has three real roots in the interval $[-2, 2]$.

Additional Questions

1. Calculate the following limits, clearly stating what rules you used:

   (a) $\lim_{x\to 8} \frac{2x^2-17x+8}{8-x}$

   (b) $\lim_{x\to 0} \frac{x}{3-\sqrt{x+9}}$

   (c) $\lim_{x\to\infty} x\sin(\frac{\pi}{x})$

   (d) $\lim_{x\to\infty} \frac{1+x}{x^x}$ (Hint: use the fact that $x^x \geq x^2$ for all $x \geq 2$.)

   Solution

(a) By factorising, $2x^2 - 17x + 8 = (2x-1)(x-8)$, so by the sum and product rules,

$$\begin{aligned}\lim_{x\to 8} \frac{2x^2-17x+8}{x-8} &= \lim_{x\to 8} \frac{(2x-1)(x-8)}{8-x} \\ &= \lim_{x\to 8} -(2x-1) \\ &= -2(\lim_{x\to 8} x) + 1 \\ &= -15\end{aligned}$$

(b) Since $(3-\sqrt{x+9})(3+\sqrt{x+9}) = 9-(x+9) = -x$,

$$\begin{aligned}\lim_{x\to 0} \frac{x}{3-\sqrt{x+9}} &= \lim_{x\to 0} \frac{x(3+\sqrt{x+9})}{(3-\sqrt{x+9})(3+\sqrt{x+9})} \\ &= \lim_{x\to 0} \frac{x(3+\sqrt{x+9})}{-x} \\ &= \lim_{x\to 0} -(3+\sqrt{x+9}) \\ &= \lim_{x\to 0} (-3-\sqrt{x+9})\end{aligned}$$

For all $x \geq 0$, $3 \leq \sqrt{x+9} \leq x+3$. Since $\lim_{x\to 0}(x+3) = (\lim_{x\to 0} x) + 3 = 3$ by the sum rule, the sandwich rule can be applied and so $\lim_{x\to 0} \sqrt{x+9} = 3$. Hence, by the sum rule,

$$\begin{aligned}\lim_{x\to 0} \frac{x}{3-\sqrt{x+9}} &= \lim_{x\to 0} (-3-\sqrt{x+9}) \\ &= -3 - (\lim_{x\to 0} \sqrt{x+9}) \\ &= -6\end{aligned}$$

(c) Let $t = \frac{1}{x}$, so $x \to \infty$ if and only if $t \to 0$. Then,

$$\begin{aligned}\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) &= \lim_{t\to 0} \frac{\sin\left(\frac{\pi}{\frac{1}{t}}\right)}{t} \\ &= \lim_{t\to 0} \frac{\sin(\pi t)}{t} \\ &= \lim_{t\to 0} \frac{\pi \sin(\pi t)}{\pi t}\end{aligned}$$

Let $s = \pi t$, so $t \to 0$ if and only if $s \to 0$. Then, using the fact that $\lim_{s\to 0} \frac{\sin(s)}{s} = 1$ with the product rule,

$$\begin{aligned}\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) &= \lim_{t\to 0} \frac{\pi \sin(\pi t)}{\pi t} \\ &= \lim_{s\to 0} \frac{\pi \sin(s)}{s} \\ &= \pi \lim_{s\to 0} \frac{\sin(s)}{s} \\ &= \pi\end{aligned}$$

(d) By definition, for all $x \geq 2$, $x^x \geq x^2$ so $0 \leq \frac{1+x}{x^x} \leq \frac{1+x}{x^2}$ for all $x \geq 2$.
Let $t = \frac{1}{x}$, so $x \to \infty$ if and only if $t \to 0$. Then, by the sum and product rules,

$$\begin{aligned}\lim_{x\to\infty} \frac{1+x}{x^2} &= \lim_{t\to 0} \frac{1+\frac{1}{t}}{\frac{1}{t^2}} \\ &= \lim_{t\to 0} (t^2+t) \\ &= (\lim_{t\to 0} t)^2 + (\lim_{t\to 0} t) \\ &= 0\end{aligned}$$

so the sandwich rule can be applied and so $\lim_{x\to\infty} \frac{1+x}{x^x} = 0$.

2. Use the following theorem to find the limits of

   (a) e^{4x+1} as $x \to -\frac{1}{2}$.

   (b) $2^{\sin(x)}$ as $x \to \frac{\pi}{2}$.

   Theorem. *If f and g are real-valued functions where $\lim_{x\to a} f(x) = L$ and g is continuous at $x = L$, then $\lim_{x\to a}(g \circ f)(x) = g(\lim_{x\to a} f(x)) = g(L)$.*

   Solution

   (a) Let $f(x) = 4x + 1$ and $g(x) = e^x$, so $e^{4x+1} = g(f(x)) = (g \circ f)(x)$.
   Since f is continuous on $\mathbb{R}$, $\lim_{x\to-\frac{1}{2}} f(x) = f(-\frac{1}{2}) = -1$.
   Since g is continuous on $\mathbb{R}$, g is continuous at $x = -1$. By the theorem above, this implies that $\lim_{x\to-\frac{1}{2}} e^{4x+1} = g(-1) = \frac{1}{e}$.

   (b) Let $f(x) = \sin(x)$ and $g(x) = 2^x$, so $2^{\sin(x)} = g(f(x)) = (g \circ f)(x)$.
   Since f is continuous on $\mathbb{R}$, $\lim_{x\to\frac{\pi}{2}} = \sin(\frac{\pi}{2}) = 1$.
   Since g is continuous on $\mathbb{R}$, g is continuous at $x = 1$. By the theorem above, this implies that $\lim_{x\to\frac{\pi}{2}} 2^{\sin(x)} = g(1) = 2$.

3. Use the definitions of convergence of sequences and continuity to show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $(a_n) \to a$, then $(f(a_n)) \to f(a)$.

   Solution Let $\varepsilon > 0$. We want to show that there is $N \in \mathbb{N}$ such that for all $n > N$, $|f(a_n) - f(a)| < \varepsilon$.

   Since f is continuous on $\mathbb{R}$, for all $b \in \mathbb{R}$, $\lim_{x\to b} f(x) = f(b)$. By definition of limits, this implies that there is $\delta > 0$ such that whenever $|x - b| < \delta$, then $|f(x) - f(b)| < \varepsilon$.

   Since $\delta > 0$, by definition of convergence, there is $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - a| < \delta$. This implies, using the definitions of continuity and limits above, that for all $n > N$, $|f(a_n) - f(a)| < \varepsilon$, so $(f(a_n)) \to f(a)$.