Differentiability

Michael Cavaliere

Week 8

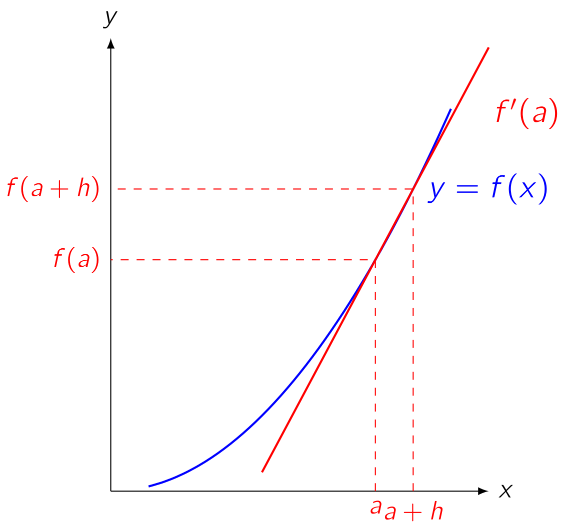
Class Content

Definition. *For a real-valued function f defined at* $a \in \mathbb{R}$*, f is* **differentiable** *at* $x = a$ *if*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The value of the limit is the **derivative** *of* $f(x)$ *at* $x = a$*, denoted* $f'(a)$*.*

The function f is **differentiable** *on an interval if it is differentiable at every point in the interval.*



Definition. *For a real-valued function f and* $a \in \mathbb{R}$*,* $f(x)$ *is* **continuous** *at* $x = a$ *if* $\lim_{x \to a} f(x) = f(a)$*.*

The function f is **continuous** *on an interval if it is continuous at every point in the interval.*

Theorem. *For a real-valued function f, if* $f(x)$ *is differentiable at* $x = a$*, then* $f(x)$ *is continuous at* $x = a$*.*

Proof. Suppose that $f(x)$ is differentiable at $x = a$, so

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Using the product rule,

$$\lim_{h \to 0} (f(a+h) - f(a)) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \times \lim_{h \to 0} h = 0$$

Let $x = a + h$, so $h = x - a$. By making this substitution,

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{h \to 0} (f(a+h) - f(a)) = 0$$

Using the sum rule,

$$\lim_{x \to a} f(x) = \lim_{x \to a} (f(x) - f(a)) + \lim_{x \to a} f(a) = f(a)$$

so $f(x)$ is continuous at $x = a$. □

Question 1

Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous but not differentiable at $x = 0$.

Theorem. *If f and g are real-valued functions that are differentiable at $x = a$, then*

- *$f(x) + g(x)$ is differentiable at $x = a$ with derivative $f'(a) + g'(a)$ (sum rule)*
- *$f(x)g(x)$ is differentiable at $x = a$ with derivative $f'(a)g(a) + f(a)g'(a)$ (product rule)*
- *$\frac{f(x)}{g(x)}$ is differentiable at $x = a$ with derivative $\frac{f'(a)g(a)-f(a)g'(a)}{(g(a))^2}$ if $g(a) \neq 0$ (quotient rule)*

Theorem (Chain Rule)**.** *If $f : A \to B$ is differentiable at $x = a$ and $g : B \to C$ is differentiable at $x = f(a)$, then $g \circ f$ is differentiable at $x = a$ with $(g \circ f)'(a) = g'(f(a))f'(a)$.*

Theorem (Leibniz' Theorem)**.** *If f and g are real-valued functions that are differentiable at $x = a$, then if $h(a) = f(a)g(a)$*

$$h^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a)$$

Question 2

1. Find $f'(x)$ where $f(x) = \frac{6x^2}{2-x}$
2. Find $f'(x)$ where $f(x) = x^x$
3. Find $f'(x)$ where $f(x) = \ln(xe^x + 1) - x^4$
4. Find $f^{(5)}(x)$ where $f(x) = x^2 \sin x$

Additional Questions

1. If $f(2) = -8$, $f'(2) = 3$, $g(2) = 17$ and $g'(2) = -4$, determine the value of $(fg)'(2)$.

2. If $f'(-1) = -2$, $g'(-1) = 0$ and $(\frac{f}{g})'(-1) = 6$, determine the value of $g(-1)$.

3. Prove from the definition that $\ln(x)$ is differentiable on $(0, \infty)$ with derivative $\frac{1}{x}$. You may assume that $\ln(x)$ is continuous on $(0, \infty)$ and that $\lim_{x \to 0}(1 + x)^{\frac{1}{x}} = e$.

4. Show that the following functions are differentiable on their domains and find their derivatives:
   (a) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \sin^6(x) + \sin(x^6)$.
   (b) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \tan^4(x^2 + 1)$.
   (c) $f : (0, \infty) \to \mathbb{R}$ where $f(x) = \frac{\ln(3x^2+1)}{3x+2}$

5. Let $f(x) = e^x x^2$. Use Leibniz' theorem to find $f^{(n)}(x)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$.