Differentiability

Michael Cavaliere

Week 8

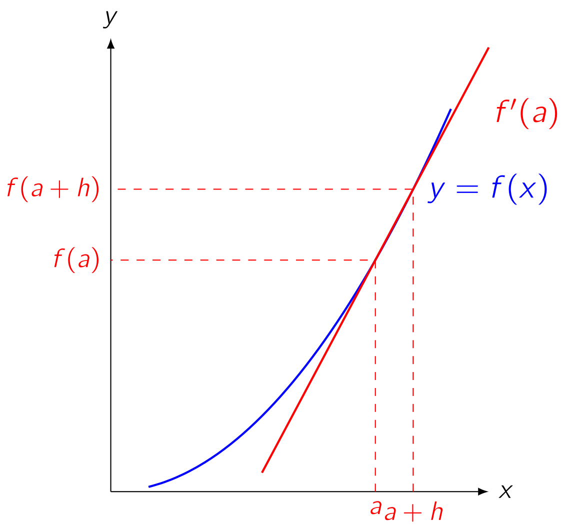
Class Content

Definition. *For a real-valued function* f *defined at* $a \in \mathbb{R}$, f *is* **differentiable** *at* $x = a$ *if*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The value of the limit is the **derivative** *of* $f(x)$ *at* $x = a$*, denoted* $f'(a)$*.*

The function f *is* **differentiable** *on an interval if it is differentiable at every point in the interval.*



Definition. *For a real-valued function* f *and* $a \in \mathbb{R}$, $f(x)$ *is* **continuous** *at* $x = a$ *if* $\lim_{x \to a} f(x) = f(a)$*.*

The function f *is* **continuous** *on an interval if it is continuous at every point in the interval.*

Theorem. *For a real-valued function* f*, if* $f(x)$ *is differentiable at* $x = a$*, then* $f(x)$ *is continuous at* $x = a$*.*

Proof. Suppose that $f(x)$ is differentiable at $x = a$, so

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Using the product rule,

$$\lim_{h \to 0} (f(a+h) - f(a)) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \times \lim_{h \to 0} h = 0$$

Let $x = a + h$, so $h = x - a$. By making this substitution,

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{h \to 0} (f(a+h) - f(a)) = 0$$

Using the sum rule,

$$\lim_{x \to a} f(x) = \lim_{x \to a} (f(x) - f(a)) + \lim_{x \to a} f(a) = f(a)$$

so $f(x)$ is continuous at $x = a$. □

Question 1

Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous but not differentiable at $x = 0$.

Solution To show $f(x)$ is continuous at $x = 0$, we need to show that $\lim_{x\to 0} f(x) = f(0) = 0$. By definition, $\lim_{x\to 0} f(x) = 0$ if for all $\varepsilon > 0$, there is $\delta > 0$ such that whenever $|x| < \delta$, $|f(x)| < \varepsilon$. For every $\varepsilon > 0$, if $\delta = \varepsilon$, then whenever $|x| < \delta$,

$$|f(x)| = ||x|| = |x| < \delta = \epsilon$$

so $f(x)$ is continuous at $x = 0$.

To show $f(x)$ is not differentiable at $x = 0$, we need to show that

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

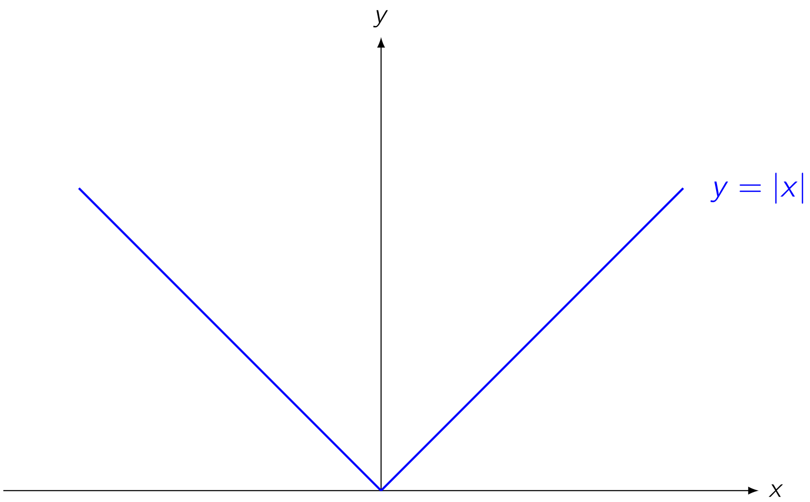
does not exist. Approaching the limit from the right, since $h > 0$,

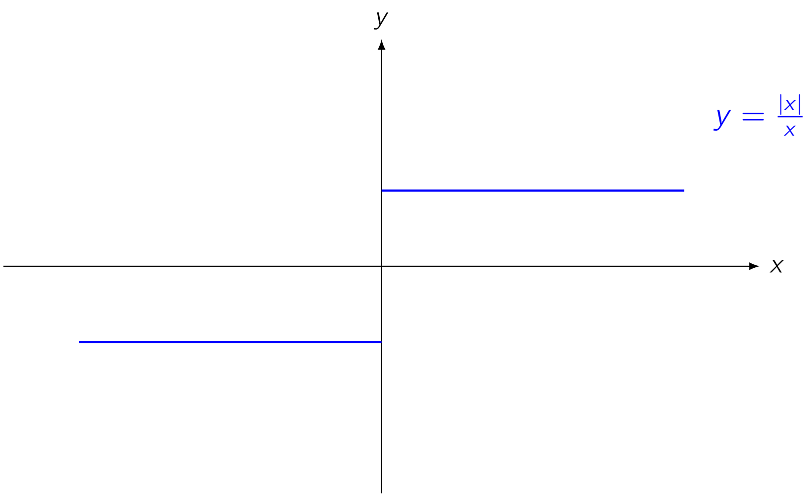
$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

Approaching the limit from the left, since $h < 0$,

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$$

Since the limits from the left and from the right are not equal, the limit cannot exist, so $f(x)$ is not differentiable at $x = 0$.





Theorem. *If f and g are real-valued functions that are differentiable at $x = a$, then*

- *$f(x) + g(x)$ is differentiable at $x = a$ with derivative $f'(a) + g'(a)$ (sum rule)*
- *$f(x)g(x)$ is differentiable at $x = a$ with derivative $f'(a)g(a) + f(a)g'(a)$ (product rule)*
- *$\frac{f(x)}{g(x)}$ is differentiable at $x = a$ with derivative $\frac{f'(a)g(a)-f(a)g'(a)}{(g(a))^2}$ if $g(a) \neq 0$ (quotient rule)*

Theorem (Chain Rule)**.** *If $f : A \to B$ is differentiable at $x = a$ and $g : B \to C$ is differentiable at $x = f(a)$, then $g \circ f$ is differentiable at $x = a$ with $(g \circ f)'(a) = g'(f(a))f'(a)$.*

Theorem (Leibniz' Theorem)**.** *If f and g are real-valued functions that are differentiable at $x = a$, then if $h(a) = f(a)g(a)$*

$$h^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)g^{(n-k)}(a)$$

Question 2

1. Find $f'(x)$ where $f(x) = \frac{6x^2}{2-x}$
2. Find $f'(x)$ where $f(x) = x^x$
3. Find $f'(x)$ where $f(x) = \ln(xe^x + 1) - x^4$
4. Find $f^{(5)}(x)$ where $f(x) = x^2 \sin x$

Solution

1. By the quotient rule,

$$f'(x) = \frac{12x(2-x) + 6x^2}{(2-x)^2} = \frac{6x(4-x)}{(2-x)^2}$$

2. By definition, $f(x) = x^x = (e^{\ln x})^x = e^{x \ln x}$. Let $g(x) = e^x$ and $h(x) = x \ln x$, so $f(x) = g(h(x))$. By the product rule,

$$h'(x) = x \times \frac{1}{x} + \ln x = 1 + \ln x$$

By the chain rule,

$$f'(x) = g'(h(x))h'(x) = e^{x \ln x}(1 + \ln x) = x^x(1 + \ln x)$$

3. Let $g(x) = \ln x$ and $h(x) = xe^x + 1$, so $g(h(x)) = \ln(xe^x + 1)$. By the product rule,

$$h'(x) = xe^x + e^x = (x + 1)e^x$$

By the chain rule,

$$(g \circ h)'(x) = \frac{(x+1)e^x}{xe^x + 1}$$

By the sum rule,

$$f'(x) = \frac{(x+1)e^x}{xe^x + 1} - 4x^3$$

4. Let $g(x) = x^2$ and $h(x) = \sin x$, so

$$\begin{aligned} g^{(1)}(x) &= 2x & h^{(1)}(x) &= \cos x \\ g^{(2)}(x) &= 2 & h^{(2)}(x) &= -\sin x \\ g^{(3)}(x) &= 0 & h^{(3)}(x) &= -\cos x \\ g^{(4)}(x) &= 0 & h^{(4)}(x) &= \sin x \\ g^{(5)}(x) &= 0 & h^{(5)}(x) &= \cos x \end{aligned}$$

By Leibniz' theorem,

$$
\begin{aligned}
f^{(5)}(x) &= g(x)h^{(5)}(x) + 5g^{(1)}(x)h^{(4)}(x) + 10g^{(2)}(x)h^{(3)}(x) \\
&\quad + 10g^{(3)}(x)h^{(2)}(x) + 5g^{(4)}(x)h^{(1)}(x) + g^{(5)}(x)h(x) \\
&= x^2\cos x + 10x\sin x - 20\cos x \\
&= (x^2 - 20)\cos x + 10x\sin x
\end{aligned}
$$

Additional Questions

1. If $f(2) = -8$, $f'(2) = 3$, $g(2) = 17$ and $g'(2) = -4$, determine the value of $(fg)'(2)$.

Solution By the product rule,

$$
\begin{aligned}
(fg)'(2) &= f(2)g'(2) + f'(2)g(2) \\
&= -8 \times -4 + 3times17 \\
&= 83
\end{aligned}
$$

2. If $f'(-1) = -2$, $g'(-1) = 0$ and $(\frac{f}{g})'(-1) = 6$, determine the value of $g(-1)$.

Solution By the quotient rule,

$$
\begin{aligned}
6 &= \left(\frac{f}{g}\right)'(-1) \\
&= \frac{f'(-1)g(-1) - f(-1)g'(-1)}{(g(-1))^2} \\
&= \frac{-2g(-1)}{(g(-1))^2} \\
&= \frac{-2}{g(-1)}
\end{aligned}
$$

so $-2g(-1) = 6$ and hence $g(-1) = -3$.

3. Prove from the definition that $\ln(x)$ is differentiable on $(0, \infty)$ with derivative $\frac{1}{x}$. You may assume that $\ln(x)$ is continuous on $(0, \infty)$ and that $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$.

Solution Using the definition of differentiability, we want to show that

$$
\lim_{h\to 0} \frac{\ln(a+h) - \ln(a)}{h} = \frac{1}{a}
$$

for all $a \in (0, \infty)$.

Using log laws,

$$\begin{aligned}\lim_{h\to 0}\frac{\ln(a+h)-\ln(a)}{h} &= \lim_{h\to 0}\frac{\ln(\frac{a+h}{a})}{h}\\ &= \lim_{h\to 0}\frac{\ln(1+\frac{h}{a})}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{a}\right)\\ &= \lim_{h\to 0}\ln\left(\left(1+\frac{h}{a}\right)^{\frac{1}{h}}\right)\end{aligned}$$

Let $t=\frac{h}{a}$, so $h\to 0$ if and only if $t\to 0$. Then, using log laws,

$$\begin{aligned}\lim_{h\to 0}\frac{\ln(a+h)-\ln(a)}{h} &= \lim_{h\to 0}\ln\left(\left(1+\frac{h}{a}\right)^{\frac{1}{h}}\right)\\ &= \lim_{t\to 0}\ln((1+t)^{\frac{1}{at}})\\ &= \lim_{t\to 0}\frac{1}{a}\ln((1+t)^{\frac{1}{t}})\end{aligned}$$

Let $f(x)=(1+x)^{\frac{1}{x}}$ and $g(x)=\ln(x)$, so $\ln((1+t)^{\frac{1}{t}})=g(f(x))=(g\circ f)(x)$. Since $\lim_{x\to 0}f(x)=e$ and g is continuous at $x=e$, $\lim_{x\to 0}(g\circ f)(x)=g(e)=1$. Hence, by the product rule,

$$\begin{aligned}\lim_{h\to 0}\frac{\ln(a+h)-\ln(a)}{h} &= \lim_{t\to 0}\frac{1}{a}\ln((1+t)^{\frac{1}{t}})\\ &= \frac{1}{a}\lim_{t\to 0}\ln((1+t)^{\frac{1}{t}})\\ &= \frac{1}{a}\end{aligned}$$

so $\ln(x)$ is continuous at $x=a$ with derivative $\frac{1}{a}$ for all $a\in(0,\infty)$.

4. Show that the following functions are differentiable on their domains and find their derivatives:
   (a) $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=\sin^6(x)+\sin(x^6)$.
   (b) $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=\tan^4(x^2+1)$.
   (c) $f:(0,\infty)\to\mathbb{R}$ where $f(x)=\frac{\ln(3x^2+1)}{3x+2}$

Solution

(a) Let $g(x)=\sin(x)$ and $h(x)=x^6$, so $f(x)=h(g(x))+g(h(x))=(h\circ g)(x)+(g\circ h)(x)$. Since g and h are differentiable on $\mathbb{R}$, by the chain rule and the sum rule, f is differentiable on $\mathbb{R}$ and

$$\begin{aligned}f'(x) = h(g(x))+g(h(x)) &= (h\circ g)'(x)+(g\circ h)'(x)\\ &= h'(g(x))g'(x)+g'(h(x))h'(x)\\ &= 6\sin^5(x)\cos(x)+6\cos(x^6)x^5\end{aligned}$$

(b) Let $u(x)=x^2+1$ and $v(x)=\tan(x)$, so $f(x)=v(u(x))=(v\circ u)(x)$.
Let $g(x)=\tan(x)$ and $h(x)=x^4$, so $v(x)=h(g(x))=(h\circ g)(x)$. Since g and h are differentiable on $\mathbb{R}$, by the chain rule, v is differentiable on $\mathbb{R}$ and

$$\begin{aligned}v'(x) &= h'(g(x))g'(x)\\ &= 4\tan^3(x)\sec^2(x)\end{aligned}$$

Since u and v are differentiable on $\mathbb{R}$, by the chain rule, f is differentiable on $\mathbb{R}$ and

$$\begin{aligned} f'(x) &= v'(u(x))u'(x) \\ &= 4\tan^3(x^2+1)\sec^2(x^2+1)(2x) \\ &= 8x\tan^3(x^2+1)\sec^2(x^2+1) \end{aligned}$$

(c) Let $u(x) = \ln(3x^2+1)$ and $v(x) = 3x+2$, so $f(x) = \frac{u(x)}{v(x)}$.
Let $g(x) = 3x^2+1$ and $h(x) = \ln(x)$, so $u(x) = h(g(x)) = (h \circ g)(x)$. Since g and h are differentiable on $(0,\infty)$, by the chain rule, u is differentiable on $(0,\infty)$ and

$$\begin{aligned} u'(x) &= h'(g(x))g'(x) \\ &= \frac{6x}{3x^2+1} \end{aligned}$$

Since u and v are differentiable on $(0,\infty)$, by the quotient rule, f is differentiable on $(0,\infty)$ and

$$\begin{aligned} f'(x) &= \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \\ &= \frac{\frac{6x}{3x^2+1}(3x+2) - 3\ln(3x^2+1)}{(3x+2)^2} \\ &= \frac{6x(3x+2) - 3(3x^2+1)\ln(3x^2+1)}{(3x+2)^2(3x^2+1)} \end{aligned}$$

5. Let $f(x) = e^x x^2$. Use Leibniz' theorem to find $f^{(n)}(x)$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Solution Let $g(x) = e^x$ and $h(x) = x^2$, so $f(x) = g(x)h(x)$. Since both g and h are n-times differentiable on $\mathbb{R}$, f is n-times differentiable on $\mathbb{R}$.

It follows that $g^{(n)}(x) = e^x$ for all $n \in \mathbb{N}$, $h'(x) = 2x$, $h''(x) = 2$ and $h^{(m)}(x) = 0$ for all $m \geq 3$.

By Leibniz' theorem, this implies that

$$\begin{aligned} f^{(n)}(x) &= \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(x)h^{(n-k)}(x) \\ &= g^{(n)}(x)h(x) + ng^{(n-1)}(x)h'(x) + \frac{n(n-1)}{2}g^{(n-2)}(x)h''(x) \\ &= e^x x^2 + 2ne^x x + n(n-1)e^x \\ &= e^x(x^2 + 2nx + n(n-1)) \end{aligned}$$