Root Finding Methods

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Week 9

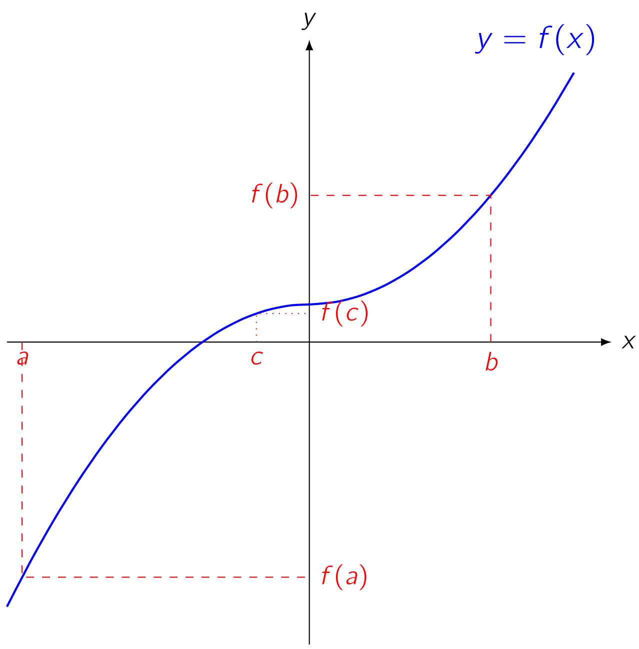
Class Content

Theorem (Intermediate Value Theorem)**.** *For a function $f : [a,b] \to \mathbb{R}$, if f is continuous on the interval $[a,b]$ with $f(a) = \alpha$ and $f(b) = \beta$, then for any $\gamma \in (\alpha, \beta)$, there is $c \in (a,b)$ such that $\gamma = f(c)$.*

The Bisection Method Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on the interval $[a,b]$ where $f(a)$ and $f(b)$ have different signs. By the intermediate value theorem, there is a root of f in (a,b). Let $c = \frac{a+b}{2}$ and consider $f(c)$.

- If $f(c) = 0$, then c is a root of f and the process stops.
- If $f(a)$ and $f(c)$ have different signs, then by the intermediate value theorem, there is a root of f in (a,c).
- Otherwise, $f(b)$ and $f(c)$ have different signs, and so by the intermediate value theorem, there is a root of f in (c,b).

Repeat this process until the value of $c \in (a,b)$ such that $f(c) = 0$ has been found.



Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 3x^7 - 5x^6 + 4x^2 - 3$.

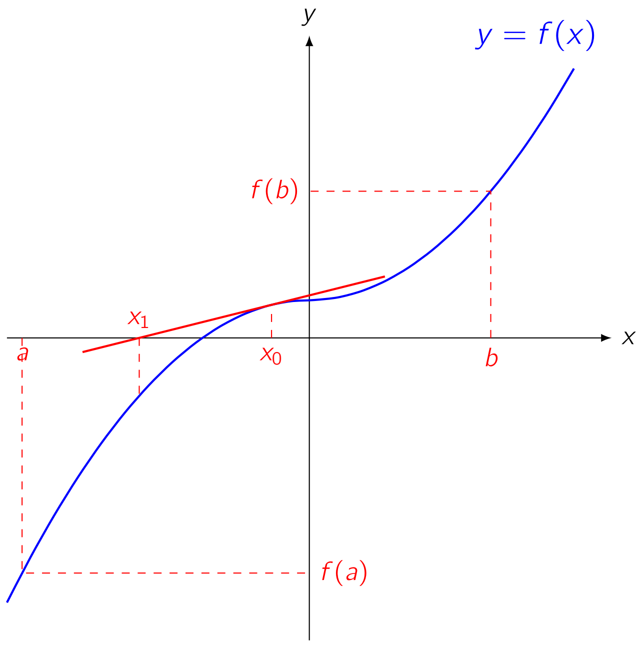
1. Find an interval $[a,b]$ for which f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have opposite signs.

2. Prove that f has a root in (a, b).
3. Use the bisection method to find this root correct to one decimal place.

The Newton-Raphson Method Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable on the interval $[a, b]$ where $f(a)$ and $f(b)$ have different signs. By the intermediate value theorem, there is a root of f in (a, b). Choose a value of $x_1 \in (a, b)$ and use the recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to converge towards the root, where x_{n+1} is the point that the tangent to the function at $(x_n, f(x_n))$ intersects the x-axis.



Question 2

Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ where $f(x) = \ln(xe^x + 1) - x^4$.

1. Find an interval $[a, b]$ for which f is differentiable on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs.
2. Prove that f has a root in (a, b).
3. Use the Newton-Raphson method to find this root correct to one decimal place.

Additional Questions

1. Use the bisection method to find a root of $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \cos x - xe^x$ to two decimal places.

2. (a) Prove that the equation $3x \ln x = 7$ has at least one solution in the interval $(2, 3)$.
   (b) Use the Newton-Raphson method to find a solution to $3x \ln x = 7$ to five decimal places.

3. Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3 + 2x^2 - 3x - 11$.

   (a) Show that for a root $x > 0$,
   $$x = \sqrt{\frac{3x + 11}{x + 2}}$$

   (b) Use direct iteration to find a root of f to three decimal places.

4. Use the intermediate value theorem and Rolle's theorem to prove that there is exactly one real root of the polynomial $f(x) = x^3 + x + 1$.

   Theorem (Rolle's Theorem)**.** *For $f : [a, b] \to \mathbb{R}$, if f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b) and $f(a) = f(b)$, then there is $c \in (a, b)$ such that $f'(c) = 0$.*