Root Finding Methods

Michael Cavaliere

Week 9

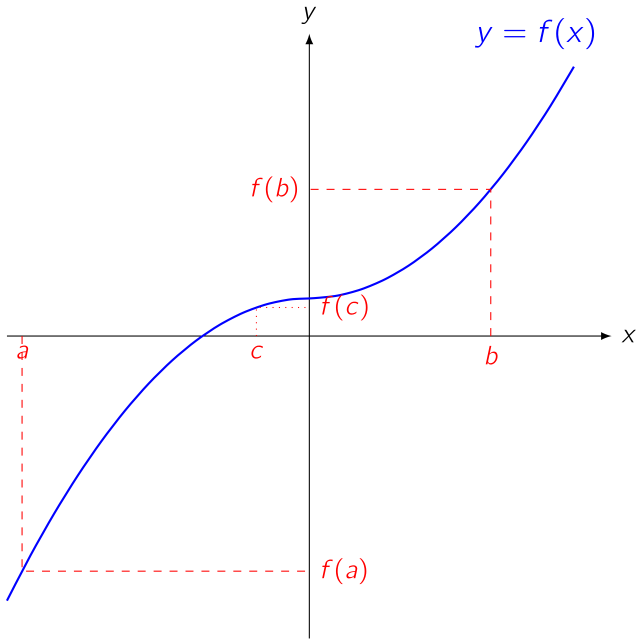
Class Content

Theorem (Intermediate Value Theorem)**.** *For a function $f : [a, b] \to \mathbb{R}$, if f is continuous on the interval $[a, b]$ with $f(a) = \alpha$ and $f(b) = \beta$, then for any $\gamma \in (\alpha, \beta)$, there is $c \in (a, b)$ such that $\gamma = f(c)$.*

The Bisection Method Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on the interval $[a, b]$ where $f(a)$ and $f(b)$ have different signs. By the intermediate value theorem, there is a root of f in (a, b). Let $c = \frac{a+b}{2}$ and consider $f(c)$.

- If $f(c) = 0$, then c is a root of f and the process stops.
- If $f(a)$ and $f(c)$ have different signs, then by the intermediate value theorem, there is a root of f in (a, c).
- Otherwise, $f(b)$ and $f(c)$ have different signs, and so by the intermediate value theorem, there is a root of f in (c, b).

Repeat this process until the value of $c \in (a, b)$ such that $f(c) = 0$ has been found.



Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 3x^7 - 5x^6 + 4x^2 - 3$.

1. Find an interval $[a, b]$ for which f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs.
2. Prove that f has a root in (a, b).
3. Use the bisection method to find this root correct to one decimal place.

Solution

1. Trying some small values:

$$f(0) = -3$$
$$f(1) = -1$$
$$f(2) = 77$$

   Since any polynomial is continuous on $\mathbb{R}$, f is continuous on the interval $[1, 2]$ and $f(1)$ and $f(2)$ have different signs.

2. Since f is continuous on the interval $[1, 2]$, by the intermediate value theorem, for any $\gamma \in (-1, 77)$, there is $c \in (1, 2)$ such that $\gamma = f(c)$. Taking $\gamma = 0$, this implies that there is a root of f in the interval $(1, 2)$.

3. Let $c_1 = \frac{1+2}{2} = 1.5$, so $f(1.5) = 0.305$ to three decimal places. Since $f(1)$ and $f(1.5)$ have different signs, the root of f must be in the interval $(1, 1.5)$.

   Let $c_2 = \frac{1+1.5}{2} = 1.25$, so $f(1.25) = -1.518$ to three decimal places. Since $f(1.25)$ and $f(1.5)$ have different signs, the root of f must be in the interval $(1.25, 1.5)$.

   Let $c_3 = \frac{1.25+1.5}{2} = 1.375$, so $f(1.375) = -1.371$ to three decimal places. Since $f(1.375)$ and $f(1.5)$ have different signs, the root of f must be in the interval $(1.375, 1.5)$.

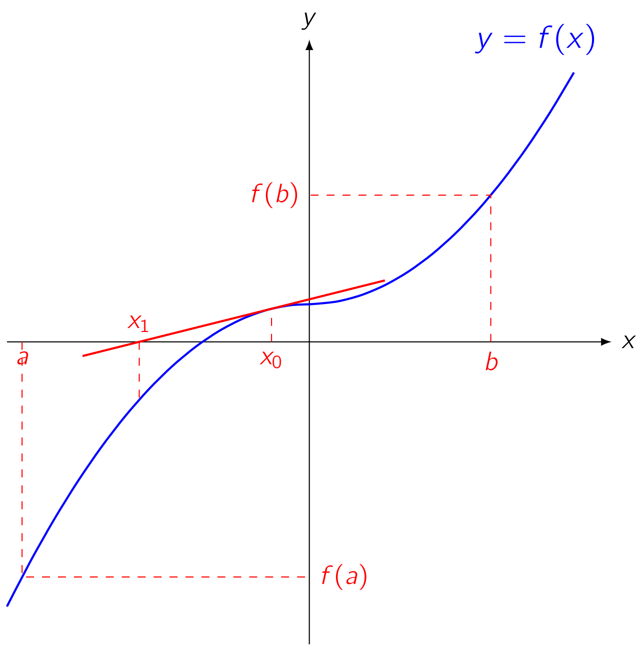
   Let $c_4 = \frac{1.375+1.5}{2} = 1.4375$, so $f(1.4375) = -0.801$ to three decimal places. Since $f(1.4375)$ and $f(1.5)$ have different signs, the root of f must be in the interval $(1.4375, 1.5)$.

   Let $c_5 = \frac{1.4375+1.5}{2} = 1.46875$, so $f(1.46875) = -0.332$ to three decimal places. Since $f(1.46875)$ and $f(1.5)$ have different signs, the root of f must be in the interval $(1.46875, 1.5)$. Hence, to one decimal place, the root of f is 1.5.

The Newton-Raphson Method Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable on the interval $[a, b]$ where $f(a)$ and $f(b)$ have different signs. By the intermediate value theorem, there is a root of f in (a, b). Choose a value of $x_1 \in (a, b)$ and use the recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to converge towards the root, where x_{n+1} is the point that the tangent to the function at $(x_n, f(x_n))$ intersects the x-axis.



Question 2

Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ where $f(x) = \ln(xe^x + 1) - x^4$.

1. Find an interval $[a, b]$ for which f is differentiable on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs.
2. Prove that f has a root in (a, b).
3. Use the Newton-Raphson method to find this root correct to one decimal place.

Solution

1. Trying some small values:

$$\begin{aligned} f(1) &= 0.313... \\ f(2) &= -13.24... \end{aligned}$$

   By the product rule, xe^x is differentiable on $\mathbb{R}_{>0}$ and so, by the sum rule, $xe^x + 1$ is differentiable on $\mathbb{R}_{>0}$. By the chain rule, $\ln(xe^x + 1)$ is differentiable on $\mathbb{R}_{>0}$. Finally, by the sum rule, f is differentiable on $\mathbb{R}_{>0}$, so f is differentiable on the interval $[a, b]$ and $f(1)$ and $f(2)$ have different signs.

2. Since f is differentiable on the interval $[1, 2]$, f is continuous on the interval $[1, 2]$. Since f is continuous on the interval $[1, 2]$, by the intermediate value theorem, for any $\gamma \in (f(2), f(1))$, there is $c \in (1, 2)$ such that $\gamma = f(c)$. Taking $\gamma = 0$, this implies that there is a root of f in the interval $(1, 2)$.

3. Using the derivative from last time,

$$f'(x) = \frac{(x+1)e^x}{xe^x + 1} - 4x^3$$

   Let $x_1 = 1.5$, so

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 1.2494... \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= 1.1298... \\ x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} \\ &= 1.1005... \\ x_5 &= x_4 - \frac{f(x_4)}{f'(x_4)} \\ &= 1.0988... \\ x_6 &= x_5 - \frac{f(x_5)}{f'(x_5)} \\ &= 1.0988... \end{aligned}$$

   Hence, to three decimal places, the root of f is 1.099.

Additional Questions

1. Use the bisection method to find a root of $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \cos x - xe^x$ to two decimal places.

Solution By the product rule, xe^x is continuous on $\mathbb{R}$, and so by the sum rule, f is continuous on $\mathbb{R}$.

Trying some small values:

$$\begin{aligned} f(0) &= 1 \\ f(1) &= -2.177... \end{aligned}$$

Since f is continuous on the interval $[0, 1]$, by the intermediate value theorem, there is $c \in (0, 1)$ such that $f(c) = 0$.

Let $c_1 = \frac{1}{2} = 0.5$, so $f(0.5) = 0.053$ to three decimal places. Since $f(0.5)$ and $f(1)$ have different signs, the root is in $(0.5, 1)$.

Let $c_2 = \frac{0.5+1}{2} = 0.75$, so $f(0.75) = -0.856$ to three decimal places. Since $f(0.5)$ and $f(0.75)$ have different signs, the root is in $(0.5, 0.75)$.

Let $c_3 = \frac{0.5+0.75}{2} = 0.625$, so $f(0.625) = -0.357$ to three decimal places. Since $f(0.5)$ and $f(0.625)$ have different signs, the root is in $(0.5, 0.625)$.

Let $c_4 = \frac{0.5+0.625}{2} = 0.5625$, so $f(0.5625) = -0.141$ to three decimal places. Since $f(0.5)$ and $f(0.5625)$ have different signs, the root is in $(0.5, 0.5625)$.

Let $c_5 = \frac{0.5+0.5625}{2} = 0.53125$, so $f(0.53125) = -0.042$ to three decimal places. Since $f(0.5)$ and $f(0.53125)$ have different signs, the root is in $(0.5, 0.53125)$.

Let $c_6 = \frac{0.5+0.53125}{2} = 0.515625$, so $f(0.515625) = 0.006$ to three decimal places. Since $f(0.515625)$ and $f(0.53125)$ have different signs, the root is in $(0.515625, 0.53125)$.

Let $c_7 = \frac{0.515625+0.53125}{2} = 0.5234375$, so $f(0.5234375) = -0.017$ to three decimal places. Since $f(0.515625)$ and $f(0.5234375)$ have different signs, the root is in $(0.515625, 0.5234375)$.

Hence, to two decimal places, the root of f is 0.52.

2. (a) Prove that the equation $3x \ln x = 7$ has at least one solution in the interval $(2, 3)$.
   (b) Use the Newton-Raphson method to find a solution to $3x \ln x = 7$ to five decimal places.

Solution

(a) Let $f(x) = 3x \ln x - 7$, so c is a solution to the equation above if and only if $f(c) = 0$. By the product rule, $3x \ln x$ is differentiable on $\mathbb{R}_{>0}$ and, by the sum rule, f is differentiable on $\mathbb{R}_{>0}$, so f is differentiable and hence continuous on the interval $[2, 3]$.

Since $f(2) = -2.841...$ and $f(3) = 2.8875...$, by the intermediate value theorem, there is $c \in (2, 3)$ such that $f(c) = 0$, and hence c is a solution to the equation above.

(b) Since f is differentiable on $[2, 3]$ and, by the product rule and the sum rule, $f'(x) = 3(\ln x + 1)$, the Newton-Raphson method can be used.

Let $x_1 = 2.5$, so

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 2.522233... \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= 2.522182... \\ x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} \\ &= 2.522182... \end{aligned}$$

Hence, to five decimal places, the solution to the equation above is 2.52218.

3. Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3 + 2x^2 - 3x - 11$.

(a) Show that for a root $x > 0$,

$$x = \sqrt{\frac{3x + 11}{x + 2}}$$

(b) Use direct iteration to find a root of f to three decimal places.

Solution

(a) Let x be a root of f with $x > 0$, so

$$\begin{aligned} x^3 + 2x^2 - 3x - 11 &= 0 \\ \implies x^3 + 2x^2 &= 3x + 11 \\ \implies x^2(x + 2) &= 3x + 11 \\ \implies x^2 &= \frac{3x + 11}{x + 2} \\ \implies x &= \sqrt{\frac{3x + 11}{x + 2}} \end{aligned}$$

(b) Let

$$x_{n+1} = \sqrt{\frac{3x_n + 11}{x_n + 2}}$$

If $x_1 = 0$, then

$$\begin{aligned} x_2 &= \sqrt{\frac{3x_1 + 11}{x_1 + 2}} = 2.3452... \\ x_3 &= \sqrt{\frac{3x_2 + 11}{x_2 + 2}} = 2.0373... \\ x_4 &= \sqrt{\frac{3x_3 + 11}{x_3 + 2}} = 2.0587... \\ x_5 &= \sqrt{\frac{3x_4 + 11}{x_4 + 2}} = 2.0571... \\ x_6 &= \sqrt{\frac{3x_5 + 11}{x_5 + 2}} = 2.0572... \end{aligned}$$

Hence, to three decimal places, the root of f is 2.057.

4. Use the intermediate value theorem and Rolle's theorem to prove that there is exactly one real root of the polynomial $f(x) = x^3 + x + 1$.

Theorem (Rolle's Theorem)**.** *For $f : [a, b] \to \mathbb{R}$, if f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b) and $f(a) = f(b)$, then there is $c \in (a, b)$ such that $f'(c) = 0$.*

Solution By definition, when $x > 0$, $f(x) > 0$, so there are no positive roots of f. When $x < -1$, $x^3 + x < -1$, so $f(x) < 0$ and so there are no roots of f less than -1. This implies that if there are any real roots of f, then they are in the interval $[-1, 0]$.

$$\begin{aligned} f(-1) &= -1 \\ f(0) &= 1 \end{aligned}$$

Since f is a polynomial, f is continuous on $[-1, 0]$ so by the intermediate value theorem, for every $\gamma \in (-1, 1)$, there is some $c \in (-1, 0)$ such that $\gamma = f(c)$. That implies that there is at least one root of f in $(-1, 0)$ by taking $\gamma = 0$.

Since f is a polynomial, f is differentiable on $(-1, 0)$. Suppose there is a second root $d \in (-1, 0)$. Since f is continuous on $[-1, 0]$ and differentiable on $(-1, 0)$, f is continuous on $[c, d] \subseteq [-1, 0]$ and differentiable on $(c, d) \subseteq (-1, 0)$. Since $f(c) = f(d)$, by Rolle's theorem, there exists $x \in (c, d)$ such that $f'(x) = 0$.

However, $f'(x) = 3x^2 + 1$ and so for any $x \in (c, d)$, $x < 0$ and so $f'(x) < 0$. This is a contradiction, so there must be exactly one real root of f.