Assignment 1: Vectors, Matrices and Coordinates

This assignment covers the first three sections of the module. In order to obtain full marks you must have a good understanding of the following topics:

Vectors Vector addition and scalar multiplication.

Matrices Matrix operations: matrix addition and multiplication, scalar multiplication, transposition. Calculation and properties of the determinant and trace. Minors, the adjoint matrix, matrix inversion. Matrix transformations, linearity, matrix representation of linear maps.

Coordinate systems Spanning, linear dependence and independence, bases for $\mathbb{R}^n$, change of basis matrices,

You are encouraged to complete online quizzes 1 and 2 before attempting this assignment.

The questions themselves are marked out of a total of 20. There are a further five marks for clarity of exposition. In particular, you should write clearly and concisely, but using full sentences where appropriate, and ensure that you explain the details of any necessary steps in your arguments. The assignment is therefore marked out of 25, and will contribute up to 5% towards your overall mark for this module.

The deadline for this assignment is: **2pm on Friday 7 February 2025 (week 5)**. You should submit your solutions via Tabula.

---

1 (a) For each of the following transformations $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, say whether f is linear or not.

(i) $f(x, y) = (0, y-3x)$ **(ii)** $f(x, y) = (x-y-1, 2x+y)$ **(iii)** $f(x, y) = (x, y)$

If it is, write down the matrix that represents f relative to the standard basis $\{\mathbf{i}, \mathbf{j}\}$. **[3]**

$\underline{i} = \begin{pmatrix}1\\0\end{pmatrix}$, $\underline{j} = \begin{pmatrix}0\\1\end{pmatrix}$

(b) Let f and g be linear transformations represented by the matrices $A = \begin{bmatrix}-2 & 0\\ 4 & -3\end{bmatrix}$ and $B = \begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}$, respectively, relative to the standard basis $\{\mathbf{i}, \mathbf{j}\}$. Determine the matrices that represent the composite transformations

i.e. $f(\underline{x}) = A\underline{x}$

$g(\underline{x}) = B\underline{x}$

(i) $g \circ f\colon \mathbb{R}^2 \to \mathbb{R}^2$ **(ii)** $f \circ g\colon \mathbb{R}^2 \to \mathbb{R}^2$

Hence find a rule for each composite transformation in the form $F(x, y) = (ax + by, cx + dy)$. **[4]**

2 Let

$$T = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}.$$

(a) Show that T is a basis for $\mathbb{R}^3$. **[4]**

(b) Calculate the change of basis matrix Q that converts from this basis T to the standard basis $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, and its inverse Q^{-1} that converts from S to T. **[3]**

$\underline{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\underline{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\underline{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

(c) Hence find the coordinates of the vector $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ relative to this new basis T. **[2]**

3 The equation of the unit circle C is given by $x^2 + y^2 = 1$. Let f be the linear transformation represented by the matrix $A = \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$.

(a) Find an equation for the image $D = f(C)$ of the unit circle under the action of this transformation. **[2]**

(b) Calculate the area enclosed by D. **[2]**

A set $\{\underline{v}_1, \dots, \underline{v}_m\} \subseteq \mathbb{R}^n$ spans $\mathbb{R}^n$ if every $\underline{v} \in \mathbb{R}^n$ can be written as a linear combination of vectors in $\{\underline{v}_1, \dots, \underline{v}_m\}$.

i.e. $\underline{v} = a_1\underline{v}_1 + \dots + a_m\underline{v}_m$ for $a_1, \dots, a_m \in \mathbb{R}$

e.g. Let $\underline{u} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\underline{v} = \begin{pmatrix}1\\-1\end{pmatrix}$. For every $\underline{w} \in \mathbb{R}^2$, $\underline{w} = \begin{pmatrix}a\\b\end{pmatrix}$ for $a, b \in \mathbb{R}$.

We want to find $x, y \in \mathbb{R}$ such that $\underline{w} = x\underline{u} + y\underline{v}$.

$$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}x\\x\end{pmatrix} + \begin{pmatrix}y\\-y\end{pmatrix} = \begin{pmatrix}x+y\\x-y\end{pmatrix}$$

We want to solve $x+y=a$, $x-y=b$ to find x and y in terms of a and b.

$x = \dfrac{a+b}{2}$, $y = \dfrac{a-b}{2}$ so $\underline{w} = \dfrac{a+b}{2}\underline{u} + \dfrac{a-b}{2}\underline{v}$ so $\{\underline{u}, \underline{v}\}$ spans $\mathbb{R}^2$.

e.g. Let $\underline{u} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\underline{v} = \begin{pmatrix}3\\2\\0\end{pmatrix}$. Let $x, y \in \mathbb{R}$.

$$x\underline{u} + y\underline{v} = \begin{pmatrix}x\\x\\0\end{pmatrix} + \begin{pmatrix}3y\\2y\\0\end{pmatrix} = \begin{pmatrix}x+3y\\x+2y\\0\end{pmatrix}$$

$$x\underline{u} + y\underline{v} = \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} 3y \\ 2y \\ 0 \end{pmatrix} = \begin{pmatrix} x+3y \\ x+2y \\ 0 \end{pmatrix}$$

Any linear combination of $\underline{u}$ and $\underline{v}$ has a zero as the z-coordinate so we cannot write $\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in terms of $\underline{u}$ and $\underline{v}$ so $\{\underline{u}, \underline{v}\}$ does not span $\mathbb{R}^3$.

2 Show whether or not the following sets of vectors span the given vector space:

(a) $\left\{\begin{bmatrix} -5 \\ -7 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix}\right\} \subset \mathbb{R}^2$ ✓

(b) $\left\{\begin{bmatrix} 8 \\ 5 \end{bmatrix}\right\} \subset \mathbb{R}^2$ ✗

(c) $\left\{\begin{bmatrix} -5 \\ -7 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \end{bmatrix}\right\} \subset \mathbb{R}^2$ ✓

(d) $\left\{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\right\} \subset \mathbb{R}^3$ ✓

$$\underline{v} = a\begin{pmatrix} -5 \\ -7 \end{pmatrix} + b\begin{pmatrix} 6 \\ 4 \end{pmatrix} + 0\begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

A set $\{\underline{v}_1, \ldots, \underline{v}_m\} \subseteq \mathbb{R}^n$ is linearly independent if whenever $a_1\underline{v}_1 + \ldots + a_m\underline{v}_m = \underline{0}$ for $a_1, \ldots, a_m \in \mathbb{R}$, then $a_1 = \ldots = a_m = 0$.

e.g. Let $\underline{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Suppose $a\underline{u} + b\underline{v} + c\underline{w} = \underline{0}$ for $a, b, c \in \mathbb{R}$.

$$\begin{pmatrix} a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} b \\ 0 \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ c \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ a+c \\ b+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We want to solve $a+b=0$, $a+c=0$, $b+c=0$ to get $a=b=c=0$.
Hence, $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly independent.

e.g. Let $\underline{u} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\underline{w} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Since $2\underline{v} - \underline{u} - \underline{w} = \underline{0}$, $\{\underline{u}, \underline{v}, \underline{w}\}$ is not linearly independent so it is linearly dependent.

2 Show whether or not the following sets of vectors span the given vector space:

(a) $\left\{\begin{bmatrix} -5 \\ -7 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix}\right\} \subset \mathbb{R}^2$ ✓

(b) $\left\{\begin{bmatrix} 8 \\ 5 \end{bmatrix}\right\} \subset \mathbb{R}^2$ ✓

(c) $\left\{\begin{bmatrix} -5 \\ -7 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \end{bmatrix}\right\} \subset \mathbb{R}^2$ ✗

(d) $\left\{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\right\} \subset \mathbb{R}^3$ ✓

3 Show whether or not each of the sets of vectors in the previous question are linearly independent.

A set $\{\underline{v}_1, \ldots, \underline{v}_m\} \subseteq \mathbb{R}^n$ is a basis if it spans $\mathbb{R}^n$ and it is linearly independent.

can write every vector in terms of it

this way of writing it is unique

If $\{\underline{v}_1, \ldots, \underline{v}_m\} \subseteq \mathbb{R}^n$ is a basis, then $m = n$.
If $\{\underline{v}_1, \ldots, \underline{v}_m\} \subseteq \mathbb{R}^n$ spans $\mathbb{R}^n$, then $m \geqslant n$.
If $\{\underline{v}_1, \ldots, \underline{v}_m\} \subseteq \mathbb{R}^n$ is linearly independent, then $m \leq n$.

5 Show that a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subset \mathbb{R}^n$ is linearly dependent if and only if one of the vectors in S can be expressed as a linear combination of the others.

Need to prove: i) If S is linearly dependent, then one of the vectors in S can be written as a linear combination of the others.

ii) If one of the vectors in S can be written as a linear combination of the others, then S is linearly dependent.

i) Suppose S is linearly dependent. This means that $a_1\underline{v}_1 + \ldots + a_m\underline{v}_m = \underline{0}$ for some $a_1, \ldots, a_m \in \mathbb{R}$ where at least one $a_i \neq 0$.

Without loss of generality, assume $a_1 \neq 0$. By rearranging,

$$\underline{v}_1 = -\frac{(a_2\underline{v}_2 + \ldots + a_m\underline{v}_m)}{a_1} = -\frac{a_2}{a_1}\underline{v}_2 - \ldots - \frac{a_m}{a_1}\underline{v}_m$$

So $\underline{v}_1$ is a linear combination of $\{\underline{v}_2, \ldots, \underline{v}_m\}$.