Change of Basis Matrix

$\begin{pmatrix}3\\4\end{pmatrix}$ has coordinates $(3,4)$ with respect to the basis $\{\underline{i}, \underline{j}\}$.

$\begin{pmatrix}3\\4\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}0\\-1\end{pmatrix}$ so $\begin{pmatrix}3\\4\end{pmatrix}$ has coordinates $(3,-1)$ with respect to the basis $\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right\}$

The change of basis matrix from $\{\underline{u}_1, \dots, \underline{u}_n\} \subseteq \mathbb{R}^n$ to $\{\underline{v}_1, \dots, \underline{v}_n\} \subseteq \mathbb{R}^n$ is the matrix where the i^{th} column is the coordinates of $\underline{u}_i$ in terms of $\underline{v}_1, \dots, \underline{v}_n$.

"Write old in terms of new"

The coordinates of a vector in terms of $\{\underline{v}_1, \dots, \underline{v}_n\}$ can be found by multiplying this matrix by the coordinates in terms of $\{\underline{u}_1, \dots, \underline{u}_n\}$.

e.g. Let $\underline{u}_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix}2\\1\end{pmatrix}$, $\underline{v}_1 = \begin{pmatrix}1\\-1\end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix}1\\-2\end{pmatrix}$, so $\underbrace{\{\underline{u}_1, \underline{u}_2\}}_{\text{old}}$ and $\underbrace{\{\underline{v}_1, \underline{v}_2\}}_{\text{new}}$ are bases.

Let $\underline{u}_1 = a\underline{v}_1 + b\underline{v}_2$.

$$\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}a\\-a\end{pmatrix} + \begin{pmatrix}b\\-2b\end{pmatrix} \Rightarrow a+b=1,\ -a-2b=2$$
$$\Rightarrow a=4,\ b=-3$$

So $\underline{u}_1$ has coordinates $(4,-3)$ in the new basis.

Let $\underline{u}_2 = a\underline{v}_1 + b\underline{v}_2$

$$\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}a\\-a\end{pmatrix} + \begin{pmatrix}b\\-2b\end{pmatrix} \Rightarrow a+b=2,\ -a-2b=1$$

$$\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}a\\-a\end{pmatrix} + \begin{pmatrix}0\\-2b\end{pmatrix} \Rightarrow a+b=2, -a-2b=1$$
$$\Rightarrow a=5, b=-3$$

So $\underline{u}_2$ has coordinates $(5,-3)$ in the new basis.

The change of basis matrix from $\{\underline{u}_1, \underline{u}_2\}$ to $\{\underline{v}_1, \underline{v}_2\}$ is

$$A = \begin{pmatrix}4 & 5\\-3 & -3\end{pmatrix}$$

To check: $\underline{u}_1$ has coordinates $(1,0)$ in $\{\underline{u}_1, \underline{u}_2\}$ and coordinates $(4,-3)$ in $\{\underline{v}_1, \underline{v}_2\}$.

$$\begin{pmatrix}4 & 5\\-3 & -3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}4\\-3\end{pmatrix}$$

$\underline{i}$ has coordinates $(-\frac{1}{3}, \frac{2}{3})$ in $\{\underline{u}_1, \underline{u}_2\}$ and coordinates $(2,-1)$ in $\{\underline{v}_1, \underline{v}_2\}$

$$\begin{pmatrix}4 & 5\\-3 & -3\end{pmatrix}\begin{pmatrix}-\frac{1}{3}\\\frac{2}{3}\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$$

The change of basis matrix from $\{\underline{v}_1, \underline{v}_2\}$ to $\{\underline{u}_1, \underline{u}_2\}$ is A^{-1}.

4 Let $S = \{\mathbf{i}, \mathbf{j}\}$ be the standard basis for $\mathbb{R}^2$, where $\mathbf{i} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathbf{j} = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, and let $T = \left\{\left[\begin{smallmatrix}-5\\-7\end{smallmatrix}\right], \left[\begin{smallmatrix}6\\4\end{smallmatrix}\right]\right\}$ be another basis for $\mathbb{R}^2$. Calculate the matrices that convert between these bases.

From T to S: $\begin{pmatrix}-5 & 6\\-7 & 4\end{pmatrix}$

(easier to go to standard basis)

From S to T: $\begin{pmatrix}-5 & 6\\-7 & 4\end{pmatrix}^{-1} = \frac{1}{22}\begin{pmatrix}4 & -6\\7 & -5\end{pmatrix}$

Geometric Interpretations of Linear Maps

8 Write down matrices representing the following geometric transformations in the plane, relative to the standard basis vectors $\mathbf{i} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathbf{j} = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$:

(a) A reflection in the line $y = -x$

(b) An anticlockwise rotation through an angle of $\frac{\pi}{3}$ about the origin.

(c) A scaling by a factor of 3 parallel to the x–axis, and a factor of $\frac{1}{2}$ parallel to the y–axis.

Rotation by θ around the origin: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Reflection in $y = (\tan\theta)x$: $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$

Scaling by a factor of a parallel to the x-axis and a factor of b parallel to the y-axis : $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

e.g. Let C be the circle with equation $x^2+y^2=4$ and let $f: \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x,y) = (x, y+2x)$.

We want an equation for $f(C)$.

$C = \{(u,v) : u^2+v^2=4\}$
$f(C) = \{f(u,v) : u^2+v^2=4\}$

Let $(x,y) \in f(C)$ so $(x,y) = f(u,v)$ for some $(u,v) \in C$.

$(x,y) = f(u,v) = (u, v+2u)$

so $x = u \quad\Rightarrow\quad u = x$
$\quad y = v+2u \qquad v = y-2x$

Since $u^2+v^2=4$, $x^2+(y-2x)^2=4$ so $5x^2-4xy+y^2=4$ is the equation for $f(C)$.

If f is a linear map with corresponding matrix A and C is a shape, then the area inside $f(C)$ is equal to $|\det(A)|$ times the area inside C.

$f: \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x,y) = (x, y+2x)$

C is a circle of radius 2 so the area inside C is 4π.

To find the area inside $f(C)$:

The matrix corresponding to f is $A = \left(f\begin{pmatrix}1\\0\end{pmatrix} \;\; f\begin{pmatrix}0\\1\end{pmatrix} \right)$

$$= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$\det(A) = 1$ so the area inside $f(C)$ is 4π.

A good way to work out geometric interpretation is to think about the unit square.