Eigenvalues and Eigenvectors

$A\underline{v} = \lambda \underline{v}$ ← eigenvector

eigenvalue

$A\underline{v} - \lambda\underline{v} = \underline{0}$
$(A - \lambda I)\underline{v} = \underline{0}$

If $\underline{v}$ is an eigenvector corresponding to λ, so is $k\underline{v}$ for $k \in \mathbb{R} - \{0\}$.

Characteristic Polynomial

$\chi(x) = \det(A - xI)$ ← Roots are eigenvalues

Diagonalisation

If A has eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding eigenvectors $\underline{v}_1, \dots, \underline{v}_n$, then $A = QDQ^{-1}$ where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \quad Q = \begin{pmatrix} \underline{v}_1 & \dots & \underline{v}_n \end{pmatrix}$$

eigenvectors are columns
same order as eigenvalues

e.g. Let $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

$A - xI = \begin{pmatrix} 2-x & 1 \\ 2 & 3-x \end{pmatrix}$

$$\begin{aligned} \chi_A(x) = \det(A - xI) &= (2-x)(3-x) - 2 \\ &= x^2 - 5x + 4 \\ &= (x-1)(x-4) \end{aligned}$$

so A has eigenvalues 1 and 4.

Consider the eigenvalue 1:

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$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 2x+y \\ 2x+3y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector for 1.

Similarly, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for 4.

Hence, $A = QDQ^{-1}$ where
$D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ (so $Q^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$)

1 Consider the following matrices:

(a) $\begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} -3 & 1 \\ 0 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} -4 & -3 \\ -5 & 4 \end{bmatrix}$ **(d)** $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

For each of these matrices:
(i) Find the characteristic polynomial $\chi_M(k)$.
(ii) Use the characteristic polynomial to find the eigenvalues and corresponding eigenvectors.
(iii) Diagonalise the matrix M: find an invertible matrix Q and diagonal matrix D such that $M = QDQ^{-1}$.

a) $D = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$
b) $D = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix}$
c) $D = \begin{pmatrix} \sqrt{31} & 0 \\ 0 & -\sqrt{31} \end{pmatrix}$, $Q = \begin{pmatrix} 3 & 3 \\ -4-\sqrt{31} & -4+\sqrt{31} \end{pmatrix}$
d) $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$

<u>Theorem</u>: If $A = QDQ^{-1}$, then $A^m = QD^mQ^{-1}$ for all $m \in \mathbb{N}$.

If $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$, then $D^m = \begin{pmatrix} \lambda_1^m & 0 & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^m \end{pmatrix}$

Linear Difference Equations

e.g. Let $x_{n+1} = 2x_n + y_n$ and $y_{n+1} = 2x_n + 3y_n$, where $x_0 = 6$ and $y_0 = 3$.

Let $\underline{z}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}}_{= T} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^n \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \underline{z}_n &= T\underline{z}_{n-1} \\ &= T^2 \underline{z}_{n-2} \\ &= T^3 \underline{z}_{n-3} \\ &= \dots \\ &= T^n \underline{z}_0 \end{aligned}$$

From before,
$$\begin{aligned} \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^n &= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^n \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4^n \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & 4^n \\ -1 & 2(4^n) \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 4^n+2 & 4^n-1 \\ 2(4^n)-2 & 2(4^n)+1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} 4^n+2 & 4^n-1 \\ 2(4^n)-2 & 2(4^n)+1 \end{pmatrix}\begin{pmatrix} 6 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 3(4^n)+3 \\ 6(4^n)-3 \end{pmatrix} \end{aligned}$$

So $x_n = 3(4^n) + 3$ and $y_n = 6(4^n) - 3$

2 Consider the following system of linear difference equations:

$$x_{n+1} = 2x_n + y_n$$
$$y_{n+1} = 3x_n + 4y_n$$

where $x_0 = y_0 = 1$. Use the matrix diagonalisation method to find a closed form solution for this system; that is, expressions for x_n and y_n in terms of n.

just in terms of n

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$, $Q^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

$x_n = \frac{5^n+1}{2}$ and $y_n = \frac{3(5^n)-1}{2}$

3 Consider the following linear second-order differential equation:

$$x_{n+1} = 3x_n - 2x_{n-1}, \qquad \text{with } x_0 = -2 \text{ and } x_1 = 1$$

(a) Rewrite this as a system of two linear first-order difference equations.

(b) Use the matrix diagonalisation method to find a closed form solution for x_n.

(c) By considering the dominant eigenvector, find the limit of the ratio of successive terms in the sequence as $n \to \infty$.

Let $y_n = x_{n-1}$. Then, $x_{n+1} = 3x_n - 2y_n$

$y_{n+1} = x_n$

with initial conditions $x_1 = 1$ and $y_1 = -2$.

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$