The leading diagonal of a matrix $A = (a_{ij})$ is the diagonal with entries $a_{11}, a_{22}, \ldots, a_{nn}$.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

$\wr$ = diagonal
$\wr + \wr$ = upper triangular

A matrix is diagonal if the only non-zero entries are on the leading diagonal.

A matix is upper triangular if the only non-zero entries are on or above the leading diagonal.

If $A\underline{v} = \lambda \underline{v}$ for $\lambda \in \mathbb{R}$ and $\underline{v} \in \mathbb{R}^n \setminus \{\underline{0}\}$, then λ is the eigenvalue corresponding to the eigenvector $\underline{v}$.

1 Let D be an $n \times n$ diagonal matrix, and let T be an $n \times n$ upper triangular matrix.

(a) Show that the eigenvalues of D are exactly the elements on the leading diagonal.

(b) Show that the eigenvalues of T are exactly the elements on the leading diagonal.

(One approach is to try the cases $n = 2$ and $n = 3$ first, then generalise.)

a) The eigenvalues of D are the roots of the characteristic polynomial $\chi_D(x) = \det(D - xI_n)$.

Let $D = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$

so $D - xI_n = \begin{pmatrix} a_{11}-x & 0 & \cdots & 0 \\ 0 & a_{22}-x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}-x \end{pmatrix}$

(can prove rigorously by induction)
so $\det(D - xI_n) = (a_{11}-x)(a_{22}-x)\ldots(a_{nn}-x)$
so the roots of χ_D are $a_{11}, a_{22}, \ldots, a_{nn}$.

Hence, the eigenvalues of D are $a_{11}, a_{22}, \ldots, a_{nn}$.

b) The eigenvalues of T are the roots of the characteristic polynomial $\chi_T(x) = \det(T - xI_n)$.

Let $T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$

so $T - xI_n = \begin{pmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-x & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}-x \end{pmatrix}$

(can be proven by induction, expand down columns)

Expanding the determinant down columns gives
$\det(T - xI_n) = (a_{11}-x)(a_{22}-x)\ldots(a_{nn}-x)$
so the roots of χ_T are $a_{11}, a_{22}, \ldots, a_{nn}$.

Hence, the eigenvalues of T are $a_{11}, a_{22}, \ldots, a_{nn}$.

Note: Prove it is true = prove all cases, not just some examples.
Prove it is false = find one counterexample (e.g. not linear if $f(\underline{0}) \neq \underline{0}$)

2 Suppose that A is an $n \times n$ matrix with eigenvalues $k_1, \ldots, k_n$ and corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(a) Show that A^2 has eigenvalues $k_1^2, \ldots, k_n^2$ and eigenvalues $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(b) More generally, show that A^m has eigenvalues $k_1^m, \ldots, k_n^m$ and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ for any positive integer m.

(c) Let $h \in \mathbb{R}$ be a nonzero scalar. Show that hA has eigenvalues $hk_1, \ldots, hk_n$ and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(d) Let $h \in \mathbb{R}$ be a nonzero scalar. Show that $(A + hI)$ has eigenvalues $(k_1 + h), \ldots, (k_n + h)$ and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

a) By definition, $A\underline{v}_i = k_i \underline{v}_i$ for all $i \in \{1, \ldots, n\}$.

We want to show $A^2 \underline{v}_i = k_i^2 \underline{v}_i$ for all $i \in \{1, \ldots, n\}$.

Multiplying by A gives $A^2\underline{v}_i = A k_i \underline{v}_i$
$= k_i A \underline{v}_i$ (since k_i is a scalar)
$= k_i(k_i \underline{v}_i)$
$= k_i^2 \underline{v}_i$

Hence, $\underline{v}_i$ is an eigenvector corresponding to the eigenvalue k_i for A^2 for all $i \in \{1, \dots, n\}$.

Ideas

b) Generalising (a),

$$A^m \underline{v}_i = A^{m-1}(A\underline{v}_i) = A^{m-1} k_i \underline{v}_i = k_i A^{m-1} v_i = \dots = k_i^m \underline{v}_i$$

c) $(hA)\underline{v}_i = h(A\underline{v}_i) = hk_i \underline{v}_i$

d) $(A + hI_n)\underline{v}_i = A\underline{v}_i + hI_n v_i = A\underline{v}_i + h\underline{v}_i = k_i\underline{v}_i + h\underline{v}_i$

$= (k_i + h)\underline{v}_i$