Assignment 2: Equivalent Matrices, Eigenvalues, Eigenvectors and Diagonalisation

This assignment covers sections 4–6 of the module. In order to obtain full marks you must have a good understanding of the following topics:

Equivalent Matrices Elementary row operations, simultaneous equations, row echelon form, reduced row echelon form, rank.

Eigenvalues and Eigenvectors Characteristic polynomial, definition and calculation of eigenvalues and eigenvectors.

Diagonalisation Matrix diagonalisation, solving difference equations, transition matrices, long-term behaviour.

You are encouraged to complete online quizzes 2 and 3 before attempting this assignment.

The questions themselves are marked out of a total of 20. There are a further five marks for clarity of exposition. In particular, you should write clearly and concisely, but using full sentences where appropriate, and ensure that you explain the details of any necessary steps in your arguments. The assignment is therefore marked out of 25, and will contribute up to 5% towards your overall mark for this module.

The deadline for this assignment is: **2pm on Friday 28 February 2025 (week 8)**. You should submit your solutions via Tabula.

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1 For each of the following systems of equations, write down the coefficient matrix A and the augmented matrix A'. Then use a sequence of elementary row operations to convert A' to reduced row echelon form. Use your answer to say whether the systems have unique solutions, infinitely many solutions, or no solutions. If the system has consistent solutions, then write down what they are. **[6]**

(a) $\begin{cases} 2x - y + z &= 5 \\ 4x + 2y + z &= 4 \\ -2x + y + 2z &= 1 \end{cases}$ **(b)** $\begin{cases} -x + y &= 2 \\ 2x - y - z &= -2 \\ x - z &= 0 \end{cases}$

2 A **(left) stochastic matrix** is one which has only non-negative ($\geqslant 0$) entries, and such that the entries in each column sum to 1. Let A be any (that is, a general) 2×2 stochastic matrix.

(a) Show that one of the eigenvalues of A is 1. **[2]**

(b) Show that a general $n\times n$ stochastic matrix also has one eigenvalue equal to 1. **[2]**

(c) What happens if the column sums are 1, as above, but the entries may be any (possibly negative) real numbers. **[1]**

Hint Consider adding all the rows together.

3 Using matrix diagonalisation, solve the following system of first order difference equations:

$$x_{n+1} = 6x_n + y_n$$
$$y_{n+1} = 5x_n + 2y_n$$

with initial conditions $x_0 = 1$ and $y_0 = -1$. **[3]**

4 Suppose that firms in a particular industry fall into one of three size categories: large, medium and small. If a firm is large one year, the probabilities that it will remain large, fall into the medium size category, or become small in the next year are, respectively, 0.7, 0.2 and 0.1.
For a firm of medium size, the corresponding probabilities are, respectively, 0.1, 0.8 and 0.1.
For a small firm, the probabilities are, respectively, 0, 0.1 and 0.9.

(a) Let L_n, M_n and S_n represent the number of firms in each category after n years have elapsed. Express the above information in the matrix equation form

$$\mathbf{Z}_{n+1} = T\mathbf{Z}_n, \quad \text{where } \mathbf{Z}_n = \begin{bmatrix} L_n \\ M_n \\ S_n \end{bmatrix}.$$

(You may wish to draw a network diagram to represent the above situation.) **[1]**

(b) Find the transition matrix T and then express it in the form QDQ^{-1}, where D is a diagonal matrix. **[3]**

(c) Suppose that the total number of firms in the industry remains fixed at 4000. By considering what happens to D^n as $n \to \infty$, determine how many firms fall into each category in the long term. **[2]**

$ax + by = c$
$dx + ey = f$

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$$

Coefficient matrix: $\begin{pmatrix} a & b \\ d & e \end{pmatrix}$

Augmented matrix: $\left(\begin{array}{cc:c} a & b & c \\ d & e & f \end{array}\right)$

E_1 Interchange the ith row and the jth row: $R_i \leftrightarrow R_j$
E_2 Multiply the ith row by a nonzero scalar k: $R_i \mapsto kR_i$
E_3 Replace the ith row by k times the jth row plus the ith row: $R_i \mapsto kR_j + R_i$

So far, vector spaces are $\mathbb{R}^n$ and scalars are $\mathbb{R}$.

Now, we think about abstract vector spaces V over a field K.

A vector space V over a field K is a set with addition and scalar multiplication such that for all $\underline{u}, \underline{v}, \underline{w} \in V$ and $h, k \in K$,

V0: $\underline{u} + \underline{v} \in V$ $*$
V1: $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
V2: $u + v = v + u$

V1: $(\underline{u}+\underline{v})+\underline{w} = \underline{u}+(\underline{v}+\underline{w})$
V2: $\underline{u}+\underline{v} = \underline{v}+\underline{u}$
V3: There is $\underline{0} \in V$ such that $\underline{v}+\underline{0} = \underline{0}+\underline{v} = \underline{v}$ *
V4: There is $-\underline{v} \in V$ such that $\underline{v}+(-\underline{v}) = (-\underline{v})+\underline{v} = \underline{0}$ *
S0: $k\underline{v} \in V$ *
S1: $h(k\underline{v}) = (hk)\underline{v}$
S2: $1\underline{v} = \underline{v}$
S3: $(h+k)\underline{v} = h\underline{v} + k\underline{v}$
S4: $k(\underline{u}+\underline{v}) = k\underline{u} + k\underline{v}$

A linear map from V to W (both vector spaces over a field K) is $f: V \to W$ such that for all $\underline{u},\underline{v} \in V$ and $k \in K$, $f(\underline{u}+\underline{v}) = f(\underline{u}) + f(\underline{v})$ and $kf(\underline{v}) = f(k\underline{v})$.

2 Let V be a vector space over a field $\mathbb{K}$. Denote by V^* the **dual space** of V: the set

$$V^* = \{f: V \to \mathbb{K}\}$$

of linear maps from V to the scalar field $\mathbb{K}$, with vector addition and scalar multiplication operations given by

$$k \cdot f(\mathbf{v}) = f(k\mathbf{v}) \quad \text{and} \quad (f+g)(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v})$$

for all $k \in \mathbb{K}$ and $\mathbf{v} \in V$. Show that V^* is a vector space over $\mathbb{K}$, by briefly checking each of the vector space axioms.

Let $d,f,g \in V^*$ (so linear maps from V to K) and $h,k \in K$.

V0: The sum of linear maps is a linear map so $f+g \in V^*$.
V1: The sum of linear maps is associative so $(d+f)+g = d+(f+g)$.
V2: The sum of linear maps is commutative so $f+g = g+f$.
V3: Let $z: V \to K$ where $z(\underline{v}) = 0$. Then, z is a linear map and $(f+z)(\underline{v}) = f(\underline{v}) + 0 = f(\underline{v})$ and similarly $(z+f)(\underline{v}) = 0 + f(\underline{v}) = f(\underline{v})$.
V4: Since f is a linear map, $-f$ is a linear map, and $(f+(-f))(\underline{v}) = f(\underline{v}) - f(\underline{v}) = 0 = z(\underline{v})$ and similarly $((-f)+f)(\underline{v}) = -f(\underline{v}) + f(\underline{v}) = 0 = z(\underline{v})$.
S0: A scalar multiple of a linear map is a linear map so $kf \in V^*$.
S1: $h(kf(\underline{v})) = h(f(k\underline{v}))$
$= f(hk\underline{v})$
$= (hk)f(\underline{v})$
S2: $1f(\underline{v}) = f(1\underline{v}) = f(\underline{v})$
S3: $(h+k)f(\underline{v}) = f((h+k)\underline{v})$
$= f(h\underline{v}+k\underline{v})$
$= f(h\underline{v}) + f(k\underline{v})$
$= hf(\underline{v}) + kf(\underline{v})$

$= f(h\underline{v} + k\underline{v})$

$= f(h\underline{v}) + f(k\underline{v})$

$= hf(\underline{v}) + kf(\underline{v})$

S4: $k(f+g)(\underline{v}) = k(f(\underline{v}) + g(\underline{v}))$

$= kf(\underline{v}) + kg(\underline{v})$

Hence, V^* is a vector space over $\mathbb{K}$.

A subspace of a vector space V is a subset $W \subseteq V$ that is a vector space with the same addition and scalar multiplication as V.

e.g. $\{(k, 0) : k \in \mathbb{R}\} \subseteq \mathbb{R}^2$ is a subspace

You only need to check V0, V3, V4 and S0.

$\underline{u}+\underline{v} \in W$ $\quad$ $\underline{0} \in W$ $\quad$ $-\underline{v} \in W$ $\quad$ $k\underline{v} \in W$ $\quad$ for $\underline{u}, \underline{v} \in W$, $k \in \mathbb{K}$

$\ker(f) = \{\underline{v} \in V : f(\underline{v}) = \underline{0}\} \subseteq V$

subset of

$\operatorname{im}(f) = \{f(\underline{v}) : \underline{v} \in V\} \subseteq W$

3 Let $f : V \to W$ be a linear map from a vector space V to a vector space W, both with scalar field $\mathbb{K}$.

(a) Show that $\ker(f)$ is a subspace of V.

(b) Show that $\operatorname{im}(f)$ is a subspace of W.

Let $\underline{u}, \underline{v} \in \ker(f)$ and $k \in \mathbb{K}$.

Since $f(\underline{u}+\underline{v}) = f(\underline{u}) + f(\underline{v}) = \underline{0} + \underline{0} = \underline{0}$, $\underline{u}+\underline{v} \in \ker(f)$.

Since $f(\underline{0}) = \underline{0}$, $\underline{0} \in \ker(f)$.

Since $f(-\underline{v}) = -f(\underline{v}) = -\underline{0} = \underline{0}$, $-\underline{v} \in \ker(f)$.

Since $f(k\underline{v}) = kf(\underline{v}) = k\underline{0} = \underline{0}$, $k\underline{v} \in \ker(f)$.

Hence, $\ker(f)$ is a subspace of V.