

Change of basis

domain codomain

Recap: $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is linear if for all $\underline{u}, \underline{v} \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$, $f(\underline{u} + \underline{v}) = f(\underline{u}) + f(\underline{v})$ and $f(\lambda \underline{v}) = \lambda f(\underline{v})$.

The matrix A corresponding to f with respect to the basis $\{\underline{u}_1, \dots, \underline{u}_m\}$ of the domain and $\{\underline{v}_1, \dots, \underline{v}_n\}$ of the codomain is the matrix where the i^{th} column is the coordinates of \underline{u}_i with respect to $\{\underline{v}_1, \dots, \underline{v}_n\}$.

Example Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $f(x, y) = (y, x)$.
Let $\{\underline{u}, \underline{v}\}$ where $\underline{u} = (1, 0)$ and $\underline{v} = (-1, 1)$ be the basis of the domain and let $\{\underline{x}, \underline{y}\}$ where $\underline{x} = (1, 1)$ and $\underline{y} = (0, 1)$ be the basis of the codomain.

$f(\underline{u}) = f(1, 0) = (0, 1) = \underline{y}$ so $f(\underline{u})$ has coordinates $(0, 1)$ in terms of $\{\underline{x}, \underline{y}\}$.

$f(\underline{v}) = f(-1, 1) = (1, -1) = \underline{x} - 2\underline{y}$ so $f(\underline{v})$ has coordinates $(1, -2)$ in terms of $\{\underline{x}, \underline{y}\}$.

Hence, the corresponding matrix is $\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$

The change of basis matrix Q from a basis $\{\underline{v}_1, \dots, \underline{v}_n\}$ of \mathbb{R}^n to the standard basis of \mathbb{R}^n is

$Q = (\underline{v}_1 \ \dots \ \underline{v}_n)$ (coordinates of old basis in terms

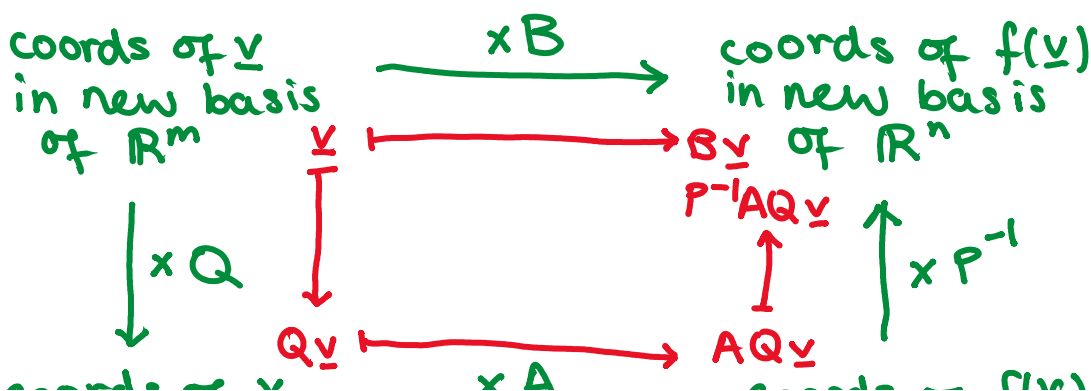
$Q = (\underline{v}_1 \dots \underline{v}_n)$ (coordinates of old basis in terms of new basis)

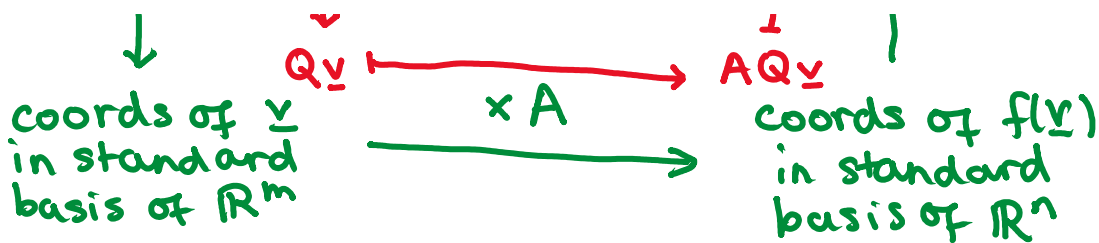
The change of basis matrix from the standard basis to $\{\underline{v}_1, \dots, \underline{v}_n\}$ is Q^{-1} .

- If
- $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear map
 - A is the $n \times m$ matrix corresponding to f with respect to the standard bases of \mathbb{R}^m and \mathbb{R}^n
 - B is the $n \times m$ matrix corresponding to f with respect to the new bases of \mathbb{R}^m and \mathbb{R}^n
 - P is the $n \times n$ change of basis matrix from the new basis of \mathbb{R}^n to the standard basis of \mathbb{R}^n
 - Q is the $m \times m$ change of basis matrix from the new basis of \mathbb{R}^m to the standard basis of \mathbb{R}^m

then $B = P^{-1}AQ$

Let $\underline{v} \in \mathbb{R}^m$, $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$





Back to example $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $f(x, y) = (y, x)$.
 New basis of domain is $\{\underline{u}, \underline{v}\}$
 where $\underline{u} = (1, 0)$ and $\underline{v} = (-1, 1)$.
 New basis of codomain is $\{\underline{x}, \underline{y}\}$
 where $\underline{x} = (1, 1)$ and $\underline{y} = (0, 1)$.

Let A be the matrix of f with respect to the standard basis, P is the change of basis matrix from $\{\underline{x}, \underline{y}\}$ to the standard basis, and Q is the change of basis matrix from $\{\underline{u}, \underline{v}\}$ to the standard basis.

$$\begin{array}{l}
 f(\underline{i}) = f(1, 0) = (0, 1) = \underline{j} \\
 f(\underline{j}) = f(0, 1) = (1, 0) = \underline{i}
 \end{array}
 \quad \left| \quad
 \begin{array}{l}
 P = (\underline{x} \quad \underline{y}) \\
 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
 Q = (\underline{u} \quad \underline{v}) \\
 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
 \end{array}$$

so $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$P^{-1}AQ = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

1 Let $S = \left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$. This is a basis for \mathbb{R}^3 .

(a) (i) Find a matrix Q that converts from S to the standard basis for \mathbb{R}^3 .

(ii) Find a matrix P that converts from the standard basis in \mathbb{R}^3 to S .

(b) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x, y, z) = (2x - 3y, 2y - 3z, 2z - 3x)$.

(i) Find a matrix A representing f relative to the standard basis in \mathbb{R}^3 .

(ii) Use your answers to part (b) to find a matrix B representing f relative to the basis S .

$$Q = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \quad Q^{-1} = \frac{1}{9} \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$$

$$f(1, 0, 0) = (2, 0, -3)$$

$$f(0, 1, 0) = (-3, 2, 0)$$

$$f(0, 0, 1) = (0, -3, 2)$$

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 0 & 2 & -3 \\ -3 & 0 & 2 \end{pmatrix}$$

$$B = Q^{-1}AQ = \begin{pmatrix} 2 & 0 & -3 \\ -3 & 2 & 0 \\ 0 & -3 & 2 \end{pmatrix}$$

Rowspace and Columnspace

If A is an $m \times n$ matrix, then

- the rowspace of A is the subspace of \mathbb{R}^n spanned by the rows of A .
- the columnspace of A is the subspace of \mathbb{R}^m spanned by the columns of A .

If R is the row reduced form of A , then

- a basis of the rowspace is the rows of R that are non-zero.
- a basis of the columnspace is the columns of A that correspond to the columns of R with a leading one.

3 Let

$$A = \begin{bmatrix} 1 & 3 & -2 & 1 & -6 \\ 1 & 4 & 4 & -2 & 5 \\ 2 & 9 & 2 & -1 & -3 \\ 3 & 13 & 6 & -5 & -2 \end{bmatrix}$$

- Find bases for the kernel, row space and column space of A .
- Verify the Dimension Theorem for A .

The row reduced form of A is
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Rowspace basis : $\{ (1, 0, 0, 0, 1), (0, 1, 0, 0, -1), (0, 0, 1, 0, 3), (0, 0, 0, 1, 2) \}$

Columnspace basis : $\{ (1, 1, 2, 3), (3, 4, 9, 13), (-2, 4, 2, 6), (1, -2, -1, -5) \}$