EC133 Week 9

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Change of basis

domain codomain

Recap: $f : \mathbb{R}^m \to \mathbb{R}^n$ is linear if for all $\underline{u}, \underline{v} \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$, $f(\underline{u} + \underline{v}) = f(\underline{u}) + f(\underline{v})$ and $f(\lambda \underline{v}) = \lambda f(\underline{v})$.

The matrix A corresponding to f with respect to the basis $\{\underline{u}_1, \dots, \underline{u}_m\}$ of the domain and $\{\underline{v}_1, \dots, \underline{v}_n\}$ of the codomain is the matrix where the i^{th} column is the coordinates of $\underline{u}_i$ with respect to $\{\underline{v}_1, \dots, \underline{v}_n\}$.

Example Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (y, x)$. Let $\{\underline{u}, \underline{v}\}$ where $\underline{u} = (1, 0)$ and $\underline{v} = (-1, 1)$ be the basis of the domain and let $\{\underline{x}, \underline{y}\}$ where $\underline{x} = (1, 1)$ and $\underline{y} = (0, 1)$ be the basis of the codomain.

$f(\underline{u}) = f(1, 0) = (0, 1) = \underline{y}$ so $f(\underline{u})$ has coordinates $(0, 1)$ in terms of $\{\underline{x}, \underline{y}\}$.

$f(\underline{v}) = f(-1, 1) = (1, -1) = \underline{x} - 2\underline{y}$ so $f(\underline{v})$ has coordinates $(1, -2)$ in terms of $\{\underline{x}, \underline{y}\}$.

Hence, the corresponding matrix is $\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$

The change of basis matrix Q from a basis $\{\underline{v}_1, \dots, \underline{v}_n\}$ of $\mathbb{R}^n$ to the standard basis of $\mathbb{R}^n$ is

$Q = (\underline{v}_1 \ \dots \ \underline{v}_n)$ (coordinates of old basis in terms

$Q = (\underline{v}_1 \;\dots\; \underline{v}_n)$ (coordinates of old basis in terms of new basis)

The change of basis matrix from the standard basis to $\{\underline{v}_1, \dots, \underline{v}_n\}$ is Q^{-1}.

If
- $f: \mathbb{R}^m \to \mathbb{R}^n$ is a linear map
- A is the $n \times m$ matrix corresponding to f with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$
- B is the $n \times m$ matrix corresponding to f with respect to the new bases of $\mathbb{R}^m$ and $\mathbb{R}^n$
- P is the $n \times n$ change of basis matrix from the new basis of $\mathbb{R}^n$ to the standard basis of $\mathbb{R}^n$
- Q is the $m \times m$ change of basis matrix from the new basis of $\mathbb{R}^m$ to the standard basis of $\mathbb{R}^m$

then $B = P^{-1}AQ$

Let $\underline{v} \in \mathbb{R}^m$, $f: \mathbb{R}^m \to \mathbb{R}^n$

coords of $\underline{v}$ in new basis of $\mathbb{R}^m$ $\xrightarrow{\times B}$ coords of $f(\underline{v})$ in new basis of $\mathbb{R}^n$

$$
\begin{tikzcd}
\underline{v} \arrow[r, mapsto] \arrow[d, "\times Q"'] & B\underline{v} \\
 & P^{-1}AQ\underline{v} \\
Q\underline{v} \arrow[r, mapsto, "\times A"'] & AQ\underline{v} \arrow[u, "\times P^{-1}"']
\end{tikzcd}
$$

$$Q\underline{v} \xmapsto{\times A} AQ\underline{v}$$

coords of $\underline{v}$ in standard basis of $\mathbb{R}^m$ $\longrightarrow$ coords of $f(\underline{v})$ in standard basis of $\mathbb{R}^n$

Back to example $f:\mathbb{R}^2 \to \mathbb{R}^2$ where $f(x,y) = (y,x)$.
New basis of domain is $\{\underline{u}, \underline{v}\}$ where $\underline{u} = (1,0)$ and $\underline{v} = (-1,1)$.
New basis of codomain is $\{\underline{x}, \underline{y}\}$ where $\underline{x} = (1,1)$ and $\underline{y} = (0,1)$.

Let A be the matrix of f with respect to the standard basis, P is the change of basis matrix from $\{\underline{x}, \underline{y}\}$ to the standard basis, and Q is the change of basis matrix from $\{\underline{u}, \underline{v}\}$ to the standard basis.

$$f(\underline{i}) = f(1,0) = (0,1) = \underline{j}$$
$$f(\underline{j}) = f(0,1) = (1,0) = \underline{i}$$

so $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$P = \begin{pmatrix} \underline{x} & \underline{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} \underline{u} & \underline{v} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$P^{-1}AQ = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

1 Let $S = \left\{ \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$. This is a basis for $\mathbb{R}^3$.

(a) **(i)** Find a matrix Q that converts from S to the standard basis for $\mathbb{R}^3$.

(ii) Find a matrix P that converts from the standard basis in $\mathbb{R}^3$ to S.

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x,y,z) = (2x - 3y, 2y - 3z, 2z - 3x)$.

(i) Find a matrix A representing f relative to the standard basis in $\mathbb{R}^3$.

(ii) Use your answers to part (b) to find a matrix B representing f relative to the basis S.

$$Q = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \qquad Q^{-1} = \frac{1}{9}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}$$

$f(1,0,0) = (2,0,-3)$
$f(0,1,0) = (-3,2,0)$
$f(0,0,1) = (0,-3,2)$

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 0 & 2 & -3 \\ -3 & 0 & 2 \end{pmatrix}$$

$$B = Q^{-1}AQ = \begin{pmatrix} 2 & 0 & -3 \\ -3 & 2 & 0 \\ 0 & -3 & 2 \end{pmatrix}$$

Rowspace and Columnspace

If A is an $m \times n$ matrix, then
- the rowspace of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.
- the columnspace of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A.

If R is the row reduced form of A, then
- a basis of the rowspace is the rows of R that are non-zero.
- a basis of the columnspace is the columns of A that correspond to the columns of R with a leading one.

3 Let

$$A = \begin{bmatrix} 1 & 3 & -2 & 1 & -6 \\ 1 & 4 & 4 & -2 & 5 \\ 2 & 9 & 2 & -1 & -3 \\ 3 & 13 & 6 & -5 & -2 \end{bmatrix}$$

(a) Find bases for the kernel, row space and column space of A.

(b) Verify the Dimension Theorem for A.

The row reduced form of A is $\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

Rowspace basis : $\{(1,0,0,0,1), (0,1,0,0,-1), (0,0,1,0,3), (0,0,0,1,2)\}$

Columnspace basis: $\{(1,1,2,3), (3,4,9,13), (-2,4,2,6), (1,-2,-1,-5)\}$