Designing controllers for unknown systems using data - lecture 1

Claudio De Persis

Institute of Engineering and Technology
J.C. Willems Center for Systems and Control

university of groningen

PhD summer school – Complex networks: dynamics and control
Mathematics Institute, University of Warwick, July 3 - 5 2019

Joint work with Pietro Tesi (Università di Firenze)

What are these lectures about?

...and how are they related to the topic of the school **Complex networks: dynamics and control**?

- Complex networks may have models too difficult to derive from first principles
- Complex networks generate large amount of data

Can we trade off the knowledge of the system's dynamics against experimental data and be able to control the system?

These lectures give a positive answer to this question for a class of systems with **simple** dynamics

Outline

Lecture 1

- **Linear control**
- **Data-based system representation**

Lecture 2

- Direct data-based control design
- Extensions

Linear control

What is control?

> *If physics is the science of understanding the physical environment, then* **control theory** *may be viewed as* **the science of modifying that environment** *[...] Control theory does not deal directly with physical reality but with* **mathematical models**.

Rudolf Kalman, Control Theory, *Encyclopædia Britannica*

Mathematical models

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x(k), u(k)) \quad k = 0, 1, 2, \ldots \end{aligned}$$

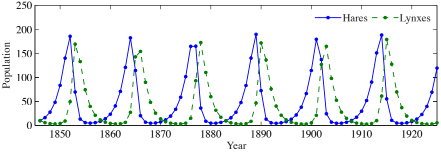
- $x(k) \in \mathbb{R}^n$ **state** of the system at **time** k (integer)
- $u(k) \in \mathbb{R}^m$ **input**
- $y(k) \in \mathbb{R}^p$ **output**
- f, h smooth mappings

Predator-prey model

$$\begin{aligned} x_1(k+1) &= x_1(k) + u(k)x_1(k) - ax_1(k)x_2(k) \\ x_2(k+1) &= x_2(k) - dx_2(k) + bx_1(k)x_2(k) \end{aligned}$$

- $x(k) \in \mathbb{R}^2$ population of prey and predator
- $u(k) \in \mathbb{R}$ birth rate per unit period controllable via food supply
- $ax_1(k)x_2(k)$ rate of predation, $bx_1(k)x_2(k)$ growth rate due to predation
- $dx_2(k)$ death rate

Population evolution



$$\begin{aligned} x(1) &= f(x(0), u(0)) \\ x(2) &= f(x(1), u(1)) = f(f(x(0), u(0)), u(1)) \\ x(3) &= f(x(2), u(2)) = f(f(f(x(0), u(0)), u(1)), u(2)) \\ &\vdots \end{aligned}$$

Aström Murray. *Feedback Systems: An Introduction for Scientists and Engineers.* Princeton University Press.

Equilibrium

A pair of constant vectors $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is an **equilibrium pair** if

$$\begin{array}{rl} x(0) = & \bar{x} \\ u(t) = & \bar{u}, \ \forall t \geq 0 \end{array} \quad \Rightarrow \quad x(t) = \bar{x}, \ \forall t \geq 0$$

where $x(t)$ is the solution to the forced difference equation

$$x(t+1) = f(x(t), \bar{u}), \quad x(0) = \bar{x}$$

An **equivalent condition** for $(\bar{x}, \bar{u})$ to be an equilibrium pair is

$$\bar{x} = f(\bar{x}, \bar{u})$$

An equilibrium of the predator-prey model

An equilibrium (not unique) of the predator-prey model is given by

$$\bar{x} = \begin{bmatrix} \frac{d}{b} \\ \frac{\bar{u}}{a} \end{bmatrix}, \quad \bar{u}$$

obtained from the solution of the system of two nonlinear algebraic equations

$$x = f(x, \bar{u}) \quad \Leftrightarrow \quad \begin{aligned} x_1 &= x_1 + \bar{u}x_1 - ax_1x_2 \\ x_2 &= x_2 - dx_2 + bx_1x_2 \end{aligned}$$

Linear control systems

A system is linear if f, h are linear in x, u

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \quad k = 0, 1, 2, \dots \end{aligned}$$

- $A \in \mathbb{R}^{n \times n}$ **dynamic** matrix
- $B \in \mathbb{R}^{n \times m}$ **control** matrix
- $C \in \mathbb{R}^{p \times n}$ **sensor** matrix
- $D \in \mathbb{R}^{p \times m}$ **direct term** matrix

Linearized predator-prey system

Example The incremental variables $x(k) := x(k, x_0) - \bar{x}$, $u(k) := u(k, x_0) - \bar{u}$ approximately satisfy [1]

$$x(k+1) \approx A\,x(k) + B\,u(k)$$

as far as x evolves in a sufficiently small neighbourhood of the equilibrium $\bar{x}$, where

$$A = \left[\frac{\partial f}{\partial x}\right]_{\bar{x},\bar{u}} = \begin{bmatrix} 1 & -\dfrac{ad}{b} \\ \dfrac{b\bar{u}}{a} & 1 \end{bmatrix}, \quad B = \left[\frac{\partial f}{\partial u}\right]_{\bar{x},\bar{u}} = \begin{bmatrix} \dfrac{d}{b} \\ 0 \end{bmatrix}$$

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[1] Taylor's expansion with order 1 around the equilibrium point $\bar{x}, \bar{u}$

$$x(k+1, x_0) = f(x(k, x_0), u(k, x_0)) = \underbrace{f(\bar{x}, \bar{u})}_{\bar{x}} + \underbrace{\left[\frac{\partial f}{\partial x}\right]_{\bar{x},\bar{u}}}_{A} \underbrace{(x(k, x_0) - \bar{x})}_{x(k)} + \underbrace{\left[\frac{\partial f}{\partial x}\right]_{\bar{x},\bar{u}}}_{B} \underbrace{(u(k, x_0) - \bar{u})}_{u(k)}$$
$$+ \underbrace{h.o.t.(x(k, x_0) - \bar{x}, u(k, x_0) - \bar{x})}_{\approx 0}$$

Solution

The solution to

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \quad k = 0, 1, 2, \ldots \end{aligned}$$

from the initial condition $x(0)$ when the input sequence $u(0), u(1), \ldots$ is applied can be recursively computed

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) \\ x(2) &= A^2x(0) + ABu(0) + Bu(1) \\ x(3) &= A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2) \\ &\vdots \end{aligned}$$

$$\begin{aligned} x(k) &= A^kx(0) + \sum_{j=0}^{k-1} A^{k-1-j}Bu(j) \\ y(k) &= CA^kx(0) + \sum_{j=0}^{k-1} CA^{k-1-j}Bu(j) + Du(k) \end{aligned}$$

Stability of an equilibrium

The stability of an equilibrium determines **whether or not unforced solutions** nearby the equilibrium **get closer to the equilibrium** as time goes by.

Definition The equilibrium $x = 0$ of the unforced system

$$x(k+1) = Ax(k)$$

is **asymptotically stable** if

- For each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\|x(0)\| < \delta(\varepsilon) \quad \Rightarrow \|x(k)\| < \varepsilon \quad \forall k \geq 0$$

- There exists $\delta > 0$ such that

$$\|x(0)\| < \delta \quad \Rightarrow \lim_{k \to +\infty} x(k) = 0$$

Stability of an equilibrium

Theorem The following are equivalent

- The equilibrium $x = 0$ of the unforced system

$$x(k+1) = Ax(k)$$

  is **asymptotically stable**

- The **spectral radius** of A is strictly smaller than 1, i.e.,

$$\rho(A) < 1$$

- There exists a unique $P \succ 0$ such that

$$A^\top PA - P \succ 0$$

```
P = dlyap(A',Q) solves the discrete-time Lyapunov equation A'PA-P=-Q, provided that A
                is asymptotically stable
```

Instability of the linearized predator-prey system

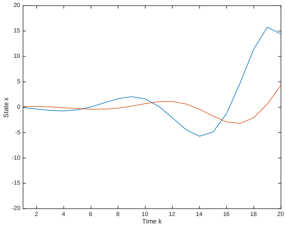
Example Let $u(k) = u(k, x_0) - \bar{u} = 0$ so that the linearized dynamics becomes

$$x(k+1) \approx A\,x(k)$$

where

$$A = \begin{bmatrix} 1 & -\dfrac{ad}{b} \\ \dfrac{b\bar{u}}{a} & 1 \end{bmatrix} \quad \Rightarrow \quad \rho(A) = \sqrt{1 + (d\bar{u})^2} > 1$$

The equilibrium is unstable: $x(0) \neq 0$ implies $x(k) \to +\infty$ as $k \to +\infty$




```
P = dlyap(A',Q)
returns no solution
```

Controllability

If an equilibrium is unstable, is it still possible to make it stable using the input? Yes, under a suitable controllability property!

Controllability is the property enjoyed by a linear system to steer any initial state to any final state by a suitable control sequence.

Definition The linear system

$$x(k+1) = Ax(k) + Bu(k)$$

is controllable if for any $x_0, x_f \in \mathbb{R}^n$ there exists an integer $T > 0$ and an input sequence

$$u(0), u(1), \ldots, u(T-1)$$

such that the solution to the linear system from the initial condition $x(0) = x_0$ under the input sequence above satisfies

$$x(T) = x_f$$

Controllability

Theorem The linear system

$$x(k+1) = Ax(k) + Bu(k)$$

is controllable if and only if the controllability matrix is full rank

$$\mathrm{rank}\left(\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}\right) = n$$

Controllability of the linearized predator-prey system

$$x(k+1) = A\,x(k) + B\,u(k)$$

where

$$A = \left[\frac{\partial f}{\partial x}\right]_{\bar{x},\bar{u}} = \begin{bmatrix} 1 & -\dfrac{ad}{b} \\ \dfrac{b\bar{u}}{a} & 1 \end{bmatrix}, \quad B = \left[\frac{\partial f}{\partial u}\right]_{\bar{x},\bar{u}} = \begin{bmatrix} \dfrac{d}{b} \\ 0 \end{bmatrix}$$

Controllability matrix

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \dfrac{d}{b} & \dfrac{d}{b} \\ 0 & \dfrac{\bar{u}d}{a} \end{bmatrix}$$

which is non singular. The system is controllable!

Stabilizing an unstable equilibrium

Theorem If the linear system

$$x(k+1) = Ax(k) + Bu(k)$$

is controllable, then for any polynomial

$$p_{des}(s) = s^n + p_1 s^{n-1} + \ldots + p_{n-1} s + p_n$$

there exists a feedback

$$u(k) = Kx(k)$$

that gives the closed-loop system

$$x(k+1) = (A + BK)x(k)$$

with characteristic polynomial equal to $p_{des}(s)$, i.e.,

$$\det(sI - A - BK) = p_{des}(s)$$

If the system is SISO, then K can be given an explicit formula

$$K = -\begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}^{-1} p_{des}(A)$$

Optimality - LQR

The **Linear Quadratic Regulation** (LQR) problem is defined as the problem of **finding the sequence of control inputs** $u(0), u(1), u(2), \dots$ that **minimizes**

$$J_\infty(x_0, u) := \sum_{k=0}^{\infty} (x(k) Q x(k) + u(k)^\top R u(k))$$

for the system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0$$

where $Q \succeq 0$ and $R \succ 0$.

Optimality - LQR

Theorem If (A, B) stabilizable and $(Q^{\frac{1}{2}}, A)$ observable[a], then

- There exists a unique control given by

$$u_\star := K_\star x, \quad K_\star := -(R + B^\top PB)^{-1} B^\top PA,$$

  with P the solution of the Discrete Algebraic Riccati Equation (DARE)

$$A^\top PA - P - A^\top PB(R + B^\top PB)^{-1} B^\top PA + Q = 0,$$

  such that the resulting cost is minimal for any x_0, that is, $J_\infty(x_0, u_\star) \leq J_\infty(x_0, u)$ for every x_0 and every u.

- $P \succ 0$, $A + BK_\star$ is asymptotically stable and the optimal cost is $x_0^\top P x_0$.

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[a] $(Q^{\frac{1}{2}}, A)$ is observable if and only if $(A^\top, Q^{\frac{1}{2}\top})$ is controllable

`[-Kstar,P,e] = dlqr(A,B,Q,R) returns the solution Kstar,P to the LQR problem`

Summary

- **Stability** of an equilibrium is a key property
- Most control problems boils down to enforce the stability of an equilibrium (**stabilization**)
- An unstable equilibrium can be **stabilized via feedback** provided that the system is controllable
- An equilibrium can be **optimally** stabilized via feedback solving the LQR problem

Stabilizing a system or solving the LQR problem requires the **exact knowledge of the system's model**, i.e., the matrices A, B

Data-based system representation

Control when the dynamics is unknown

When (A, B) are unknown, one can follow 2 distinct approaches

- **System identification** from data **+** **control** of the identified system
- **Direct** data-based control design (no identification)

These approaches require 2 different data-based representations of the unknown dynamics

- open-loop
- closed-loop

both of which are obtained using **persistently exciting** input probing signals

Persistence of excitation

Notation Given a signal $u : \mathbb{Z} \to \mathbb{R}^m$, its restriction to the interval $[k, k+T]$ in vectorized form is

$$u_{[k,k+T]} = \begin{bmatrix} u(k) \\ \vdots \\ u(k+T) \end{bmatrix}$$

Definition The signal $u_{[0,T-1]}$ is persistently exciting (PE) of order L if the Hankel matrix associated to it

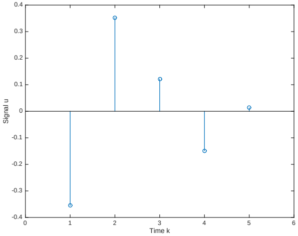
$$U_{0,L,T-L+1} = \begin{bmatrix} u(0) & u(1) & \dots & u(T-L) \\ u(1) & u(2) & \dots & u(T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(L-1) & u(L) & \dots & u(T-1) \end{bmatrix}$$

has full rank mL.

PE requires sufficiently long probing input sequences: $T \geq L(m+1) - 1$

Code to generate PE signals

```
global L m T ud

% Initializing the length of the probing input sequence

T=L*(m+1)-1;

% Generating the probing input sequence ud on [0,T-1]
% taking values in the interval [-0.5,0.5] in the form
% of an m x T matrix [ud(0) ud(1) .... ud(T-1)]

aux=zeros(m,T);
aux(:)=0.5;
ud(1:m,1:T)=rand(m,T)-aux;

% Computing the Hankel matrix Ud on [0,T-1]

for j=1:T-L+1
    for i=1:L
        Ud((i-1)*m+1:(i-1)*m+m,j)=ud(1:m, j+i-1);
    end
end

% If rank(Ud)= m*L then the sequence ud(0),...ud(T-1) is PE of order L as
% desired

if rank(Ud)== m*L
disp('input sequence is PE');
end
```



$L = 3,\ n = 2,\ m = 1 \Rightarrow T = 5$

$u_{[0,T-1]} = [-0.355\ 0.353\ 0.1221\ \ -0.149\ 0.0132]$

$$U_{0,L,T-L+1} = \begin{bmatrix} -0.3550 & 0.3530 & 0.1221 \\ 0.3530 & 0.1221 & -0.1490 \\ 0.1221 & -0.1490 & 0.0132 \end{bmatrix}$$

A fundamental lemma

A PE input applied to a controllable system produces data that are sufficiently independent over time

Lemma Let system

$$x(k+1) = Ax(k) + Bu(k)$$

be controllable. Then

$$u_{[0,T-1]} \text{ PE of order } n+t \quad \Rightarrow \quad \text{rank} \begin{bmatrix} U_{0,t,T-t+1} \\ X_{0,T-t+1} \end{bmatrix} = n + tm$$

$$U_{0,t,T-t+1} = \begin{bmatrix} u(0) & u(1) & \dots & u(T-t) \\ u(1) & u(2) & \dots & u(T-t+1) \\ u(2) & u(3) & \dots & u(T-t+2) \\ \vdots & & & \\ u(t-1) & u(t) & \dots & u(T-1) \end{bmatrix} \quad X_{0,T-t+1} = \begin{bmatrix} x(0) & x(1) & \dots & x(T-t) \end{bmatrix}$$

J.C. Willems, P. Rapisarda, I. Markovsky, B.L. De Moor. "A note on persistency of excitation." Systems & Control Letters, 54, 4, 325–329, 2005.

Example

Linearized predator-prey model $(n = 2,\ m = 1)$

$u_{[0,T-1]}$ PE of order $n + t = 3$ $(n = 2,\ t = 1)$, with $T = (n + t)(m + 1) - 1 = 5$

$$u_{[0,T-1]} = \begin{bmatrix} -0.3550 & 0.3530 & 0.1221 & -0.1490 & 0.0132 \end{bmatrix}$$

We "experimentally" compute the matrix

$$\left[\begin{array}{c} U_{0,t,T-t+1} \\ \hline X_{0,T-t+1} \end{array}\right] = \left[\begin{array}{c} U_{0,1,5} \\ \hline X_{0,5} \end{array}\right] = \left[\begin{array}{ccccc} -0.3550 & 0.3530 & 0.1221 & -0.1490 & 0.0132 \\ \hline 0.4027 & 0.3478 & 0.3571 & 0.3216 & 0.2362 \\ 0.4448 & 1.1451 & 1.7499 & 2.3708 & 2.9301 \end{array}\right]$$

where

$$X_{0,5} = \begin{bmatrix} x(0) & x(1) & x(2) & x(3) & x(4) \end{bmatrix}$$

contains the state response of the system from the initial condition $x(0)$ to the input $u_{[0,4]}$, **which has rank** $n + tm = 3$.

Open-loop data-based representation

Experiment

- Consider a PE input $u_{[0,T-1]}$ of order $n+t$ with $t=1$
- Apply it to the system

$$x(k+1) = Ax(k) + Bu(k) \quad (\star)$$

  and collect the state response in the $n \times T$ matrices

$$\begin{aligned} X_{0,T} &= \begin{bmatrix} x(0) & x(1) & \dots & x(T-1) \end{bmatrix} \\ X_{1,T} &= \begin{bmatrix} x(1) & x(2) & \dots & x(T) \end{bmatrix} \end{aligned}$$

- By the Fundamental Lemma

$$\text{rank} \begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix} = n + m$$

Theorem System $(\star)$ has the equivalent representation

$$x(k+1) = X_{1,T} \begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix}^{\dagger} \begin{bmatrix} u(k) \\ x(k) \end{bmatrix}$$

Open-loop data-based representation

Proof Let

$$S := \begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix}.$$

By Rouché-Capelli's theorem, for any given pair (u, x), the system

$$\begin{bmatrix} u \\ x \end{bmatrix} = Sg$$

admits infinite solutions g, given by

$$g = S^{\dagger} \begin{bmatrix} u \\ x \end{bmatrix} + \Pi_S^{\perp} w, \quad w \in \mathbb{R}^T,$$

where $\Pi_S^{\perp} := \left(I - S^{\dagger} S\right)$ is the orthogonal projector onto the kernel of S.

Hence,

$$x(k+1) = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} u(k) \\ x(k) \end{bmatrix} = \begin{bmatrix} B & A \end{bmatrix} Sg(k), \quad \text{for some } g(k)$$

Note that $\begin{bmatrix} B & A \end{bmatrix} S = AX_{0,T} + BU_{0,1,T} = X_{1,T}$.

Overall, we thus have

$$x(k+1) = X_{1,T} \left(S^{\dagger} \begin{bmatrix} u(k) \\ x(k) \end{bmatrix} + \Pi_S^{\perp} w \right) = X_{1,T} S^{\dagger} \begin{bmatrix} u(k) \\ x(k) \end{bmatrix}, \quad \text{since } X_{1,T} \Pi_S^{\perp} = \begin{bmatrix} B & A \end{bmatrix} S \Pi_S^{\perp} = 0$$

This concludes the proof.

Remarks

- The least square problem

$$\min_{[B\ \ A]} \left\| X_{1,T} - \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix} \right\|_{\mathrm{F}}$$

  has the solution

$$[B\ \ A] = X_{1,T} \begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix}^{\dagger}$$

- Performing the SVD of

$$\begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix} = U_1 \Sigma V_1^{\top}$$

  one obtains

$$[B\ \ A] = X_{1,T} \begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix}^{\dagger} = X_{1,T} V_1 \Sigma^{-1} U_1^{\top}$$

  which is the popular "Dynamic mode decomposition" used to estimate A, B from data.
- The open-loop data-based model of the system can be used for any control design.
- The result belongs to the class of subspace identification modes

J.L. Proctor, S.L. Brunton, J.N. Kutz. "Dynamic mode decomposition with control." *SIAM Journal on Applied Dynamical Systems*, 15, 1, 142–161, 2016.
M. Verhaegen, V. Verdult. *Filtering and system identification: a least squares approach*. Cambridge University Press, 2007.