Question 1

Consider the following statements and for each one state whether it is TRUE or FALSE. If you answer TRUE, no proof or counterexample is required. If you answer FALSE, you should give a counterexample.

a) Every decreasing sequence is bounded above. [2]
TRUE
b) If $(a_n) \to 0$ then $(\frac{1}{a_n}) \to \infty$. [2]
FALSE $a_n = -\frac{1}{n}$, $(a_n) \to \infty$ and $(\frac{1}{a_n}) \to -\infty$
c) If $a_n^2 \geq 1$ for all n and $a_n \to a$ then $a \geq 1$ [2]
FALSE $a_n = -1$, $a = -1$ but $a_n^2 = 1$
d) Every bounded sequence is Cauchy. [2]
FALSE $a_n = (-1)^n$ is bounded by 1 and -1, but is not Cauchy
e) If $\lim(a_{2n}) = \lim(a_{2n+7})$ then (a_n) converges. [2]
TRUE
f) If A is a bounded non-empty set of real numbers such that $a < 1$ for all $a \in A$ then $\inf A < 1$ [2]
TRUE
g) If $a_n \geq 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n^2$ converges [2]
TRUE
h) Every increasing sequence bounded above is Cauchy [2]
TRUE
i) If $a_n \geq 0$ for all n and $\frac{a_{n+1}}{a_n} \to \frac{1}{2}$ then $\sum_{n=1}^{\infty} a_n$ converges. [2]
TRUE
j) The sequence $\sin(n!)$ has a Cauchy subsequence. [2]
TRUE

Question 2

a) If A is a non-empty set of real numbers and U is a real number, give the definition of:
U is an upper bound of A

$$\forall a \in A, a \leq U$$

A is bounded above

$$\exists U \in \mathbb{R} \text{ such that } U \text{ is an upper bound of } A$$

U is the least upper bound of A [3]

U is an upper bound of A, and if V is any other upper bound of A then $U \leq V$.
State the completeness property in terms of the above concepts. [3]

Any set bounded above has a least upper bound.
Find the least upper bound of the following sets:
(i) $\mathbb{Q} \cap [0, 1]$, where $\mathbb{Q}$ is the set of rational numbers

$$1$$

(ii) $\{x \in \mathbb{R} : x^2 < 4\}$

$$2$$

(iii) $\{x \in \mathbb{R} : x \notin \mathbb{Q}, x^2 < 4\}$ [3]

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b) Prove Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \text{ for } n \geq 1 \text{ and } x \geq -1$$

[4]

Proof by induction.
Case $n = 1$:

$$(1+x)^n = (1+x)^1 = 1 + x = 1 + 1.x$$

Suppose true for $n = k$:

$$(1+x)^k \geq 1 + kx$$

Prove for $n = k + 1$:

$$\begin{aligned}(1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) \text{ by inductive hypothesis} \\ &= 1 + x + kx + kx^2 \\ &\geq 1 + (k+1)x \text{ since } x^2 \geq 0\end{aligned}$$

So by the principle of mathematical induction, this holds for all n.
Use Bernoulli's inequality to show that $(\lambda^n) \to \infty$ when $\lambda > 1$. [3]

Let $x = \lambda - 1$. Then $x > 0$.
By Bernoulli's inequality:

$$\begin{aligned}\lambda^n &= (1+x)^n \\ &\geq 1 + nx\end{aligned}$$

Since $1 + nx \to \infty$ as $n \to \infty$, we have $\lambda^n \to \infty$.
Write down the limits of the following sequences. You do not have to give proofs.

(i) $\left(\frac{n^2}{2^n}\right)$

0

(ii) $\left(\frac{2^n}{n!}\right)$

0

(iii) $\left(\frac{n!}{n^n}\sqrt[n]{n}\right)$

0

(iv) $(\sqrt[n]{2n})$

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