Analysis I - Monday 31st October 2016
Exam Style Questions - Solutions

(a) Define what it means for a sequence (a_n) to converge to a limit a.

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } |a_n - a| < \varepsilon \text{ whenever } n > N.$$

Define what it means for the sequence to be bounded.

$$\exists U, L \in \mathbb{R} \text{ such that } L \leq a_n \leq U \text{ for } n \in \mathbb{N}$$

Prove that every convergent sequence is bounded.

Let (a_n) be a convergent sequence with limit a. Let $\varepsilon = 1$. Then there exists some $N \in \mathbb{N}$ with $|a_n - a| < \varepsilon$ for all $n > N$. Let $U = \max\{a_1, \ldots, a_N, a+1\}$ and $L = \min\{a_1, \ldots, a_N, a-1\}$. Then $L \leq a_n \leq U$ for all $n \in \mathbb{N}$.

(b) Which of the following sequnces converge? Justify your answers.
Find the limit for those sequences which converge to a real number.

(i) $a_n = \frac{\sin(n)+\cos(n)}{\log n}$, for $n \geq 2$.

Note that $|\sin(n)| < 1$, $|\cos(n)| < 1$, but $\log(n) \to \infty$ as $n \to \infty$, so we have

$$\frac{-2}{\log(n)} \leq a_n \leq \frac{2}{\log(n)}$$

And so by the sandwich rule we have $(a_n) \to 0$.

(ii) $b_n = \frac{\log(n)}{3+\sin(n)+\cos(n)}$, for $n \geq 2$

As before, $|\sin(n)| < 1$ and $|\cos(n)| < 1$, and $\log(n) \to \infty$ as $n \to \infty$. So we have

$$b_n > \frac{\log(n)}{3+1+1}$$

and since $\frac{\log(n)}{5} \to \infty$ we have $(b_n) \to \infty$.

(iii) $c_n = \frac{\sin(n)+2\log(n)}{\cos(n)+\log(n)}$, for $n \geq 2$

Note that we have

$$c_n = \frac{\frac{\sin(n)}{\log(n)} + 2}{\frac{\cos(n)}{\log(n)} + 1}$$

By the quotient and sum rules, we have $(c_n) \to \frac{0+2}{0+1} = 2$.

(c) Given a sequence (a_n), recall that the nth term a_n is called a *floor term* if for every $m > n$ we have that $a_m \geq a_n$.

Using this notation, show that every sequence has either an increasing or a strictly decreasing subsequence.

Let (a_n) be a sequence. Let A be the set of all floor terms of (a_n).

If $|A| = \infty$ then let (a_{n_k}) be the sequence of floor terms. By the definition of floor terms, this must be an increasing sequence.

If $|A| < \infty$ and $A \neq \emptyset$ then let a_m be the last floor term; define $a_{n_1} = a_{m+1}$. If $A = \emptyset$ then define $a_{n_1} = a_1$. Since a_{n_1} is not a floor term, there must be some $n_2 > n_1$ such that $a_{n_2} < a_{n_1}$. Proceding inductively, for each $k \in \mathbb{N}$ we can obtain $n_{k+1} > n_k$ with $a_{n_{k+1}} < a_{n_k}$. This subsequence a_{n_k} is therefore a strictly decreasing sequence.