$G^{ab} \cong H_T^{-2}(G,\mathbb{Z}) \cong H_T^0(G,L^\times) = K^\times/N_m(L^\times)$ Cohomological approach:

$G^{ab} \cong H_T^{-2}(G,\mathbb{Z}) \cong H_T^0(G,L^\times)$

CFT Preparation

Local CFT: $\{\text{ab ext}/K\} \xleftrightarrow{1:1} \{\text{finite index subgps } K^\times \supseteq N_{L/K}(L^\times)\}$

Last time: Global CFT in terms of ideals, etc.

Earlier: Local CFT

Thm (Artin Reciprocity): L/K fin ab ext, S primes ramif in L.

Then Artin map admits a modulus $\tilde{m}$ with $S(\tilde{m}) \supseteq S$

$$I_K^{S(\tilde{m})}/i(K_{\tilde{m},1})\, N_m(I_L^{S(\tilde{m})}) \xrightarrow{\sim} \mathrm{Gal}(L/K)$$

Thm (Takagi existence): Every congruence $i(K_{m,1}) \subset H \subset I_K^m$ is of the form $H = i(K_{m,1}) \cdot N_{L/K}(I_L^m)$ for some $L_H \supset K$ (class field; finite abelian, ramified at primes dividing m)

$$I^{S(m)}/H \cong \mathrm{Gal}(L/K)$$

For $H = i(K_{m,1})$, $C_m \xrightarrow[\sim]{A} \mathrm{Gal}(L_m/K)$

$$\{\text{Abelian ext } K \subseteq L \subseteq L_m\} \xleftrightarrow{1:1} \{\text{Subgps of } C_m\}$$

$L \mapsto N_m(L^\times)$

$\mathrm{Gal}(L_H/K) \cong I^{S(m)}/H$ $\quad L_H$ class field of $H \longleftarrow H$

$L_m \longleftarrow \{1\}$

Today Main thms in terms of idèles, unites global CFT with LCFT

Reference V.4 & V.5 Milne's notes

§ Idèles and basic properties

$$C_m \xrightarrow{\sim} \mathrm{Gal}(L_m/K)$$

$$\varprojlim_m C_m \xrightarrow{\sim} \varprojlim_m \mathrm{Gal}(L_m/K) = \mathrm{Gal}(K^{ab}/K)$$

Replace by more natural topological gp: idèle class group C

Idea is to take all $K_v^\times$'s together but with locally compact topology

Def Group of Idèles $\mathbb{I}_K = \mathbb{A}_K^\times = \{(a_v) \in \prod K_v^\times \mid a_v \in \mathcal{O}_{K_v} \text{ almost all } v\}$

I_K is not a topological group with induced adelic topology

Topology on I_K: basis are sets $\prod V_v$ $\quad V_v \subset K_v^\times$ open, $V_v = \mathcal{O}_v^\times$ almost all v

~~Rem~~ Fact Surjective hom $\mathbb{I}_K \longrightarrow I_K$

$(a_v)_v \longmapsto \prod_{v \text{ finite}} p_v^{\,ord_p(a_v)}$

→ thickening/refinement of ideals

~~Idea~~ $\mathbb{I}_K$ also distinguishes between units in v finite and v infinite

i.e. between a and $-a$

~~Rem Surjective hom~~

Def idèle class group $C_K = \mathbb{I}/K^\times$

$K^\times$ is embedded through $a \mapsto (a, a, \dots)$ into $\mathbb{I}$

~~Def content of $a \in \mathbb{I}$ $\quad c: \mathbb{I} \to \mathbb{R}_{>0}$, $a \mapsto \prod |a_v|_v$~~

~~$\mathbb{I}^1 = \ker(c)$~~

§ Ray class groups as quotients of $\mathbb{I}$

For m modulus def $W_m(p) = \begin{cases} \mathbb{R}_{>0} & p \text{ real} \\ 1 + \hat{p}^{\,m(p)} & p \text{ finite} \end{cases}$

neighborhood of 1

Def $\mathbb{I}_m = \left(\prod_{p \nmid m} K_p^\times \times \prod_{p \mid m} W_m(p) \right) \cap \mathbb{I}$ $\quad$ $a_p \in K_p^\times \ \forall p$, $a_p \in \mathcal{O}_p^\times$ almost all p, $a_p \in W_m(p)$ $p \mid m$

$\mathbb{I}_J^m$

$W_m = \prod_{\substack{p \nmid m \\ p \text{ inf}}} K_p^\times \times \prod_{p \mid m} W_m(p) \times \prod_{\substack{p \nmid m \\ \text{fin}}} U_p$ $\quad$ $a_p \in K_p^\times$ inf p, $a_p \in \mathcal{O}_p^\times$ fin p, $a_p \in W_m(p)$ $p \mid m$

$K_{m,1} = K^\times \cap \prod_{p \mid m} W_m(p)$ $\quad$ (int inside $\prod_{p \mid m} K_p^\times$)

$\quad = K^\times \cap \mathbb{I}_m$ $\quad$ (inside $\mathbb{I}$)

Prop 4.6 m modulus of K.

(a) $\mathbb{I}_m \to I^{S(m)}$ gives isom $\mathbb{I}_m / K_{m,1} W_m \xrightarrow{\sim} C_m$

(b) $\mathbb{I}_m \hookrightarrow \mathbb{I}$ induces isom $\mathbb{I}_m / K_{m,1} \xrightarrow{\sim} \mathbb{I}/K^\times$

§ Characters ~~of ideals~~ idèles ~~and norms~~

~~Def~~ Recall: G finite abelian gp. $\psi : I^S \to G$ admits a modulus m ~~with support in S~~

if m has support in S and $\psi(i(K_{m,1})) = 1$

$\iff \psi$ factors through rayclass group

Artin showed $\psi_{L/K} : I^S \to \mathrm{Gal}(L/K)$ admits a modulus,

translate this to idèles

Prop 4.7 If $\psi : I^S \to G$ admits a modulus, then $\exists ! \, \phi : \mathbb{I} \to G$ of topgps s.t.

(a) ϕ is continuous

(b) $\phi(K^\times) = 1$

(c) $\phi(a) = \psi(\mathrm{id}(a)) \; \forall a \in \mathbb{I}^S := \{a \mid a_v = 1 \; \forall v \in S\}$

Every continuous $\phi : \mathbb{I} \to G$ arises from a ψ.

Pf

$$\begin{tikzcd}
 & I^m \arrow[r, "\text{admits a modulus}"] & C_m \arrow[r, "\psi"] & G \\
 & \mathbb{I}_m/K_{m,1} \arrow[r] \arrow[d, "\cong"] & \mathbb{I}_m/K_{m,1}W_m \arrow[u, "\cong"'] & \\
\mathbb{I} \arrow[r] \arrow[uurrr, bend right, "\phi"'] & \mathbb{I}/K^\times & &
\end{tikzcd}$$

In practice, characterize ~~$\mathbb{I}$~~ ϕ by (a), (b), (c) instead of diagram

$m = p \cdot \infty, \ S = \{p, \infty\}$

<u>Example</u> $K = \mathbb{Q}, \ L = \mathbb{Q}[\zeta_p]$ ψ is Artin map

$$I^S \longrightarrow C_m = (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathrm{Gal}(L/\mathbb{Q})$$

$$(R/S) \longmapsto RS^{-1}, \quad m \longmapsto [\zeta \mapsto \zeta^m]$$

Now determine corresponding $\phi : \mathbb{I} \to \mathrm{Gal}(L/\mathbb{Q})$ completely

$(a_\infty, a_2, \cdots, a_p, \cdots a_\ell) \in \mathbb{I}_\mathbb{Q}$ — if $a_\infty = a_p = 1$ then ~~to~~ same $\phi(a) = \psi(id(a))$

(c) $\phi(a) = \phi(id(a)) = [\zeta \mapsto \zeta^m]$ ~~therefore~~ $m = \prod \ell^{\mathrm{ord}_\ell(a_\ell)}$

$\phi\mathfrak{a}$

· $\tilde{p} = (1, \cdots 1, \underset{\text{place } p}{p}, 1, \cdots)$ $\quad \phi(\tilde{p}) = \phi(\tilde{p}/p)\,\phi(p) = 1$

$=1$ not hard to show using continuity: $\phi^{-1}(1)$ is some open around 1. — principal factors through class gp

· $(\tilde{a}) = (1, \cdots, 1, u, 1, \cdots)$ $u \in \mathbb{Z}_p^\times$ · some truncation $c \in \mathbb{Z}$ of $u^{-1} \in \mathbb{Z}_p$ will give $\phi(a)(\zeta) : \zeta \mapsto \zeta^c$

· $\hat{a} = (-1, 1, \cdots \underset{p}{1}, \cdots)$ $\quad -\tilde{a} = (1, -1, \cdots, -1, \underset{p}{1}, \cdots)(1, \cdots, 1, \underset{p}{-1}, 1, \cdots)$ — $a_\infty = a_p = 1$ — above

$\phi(\tilde{a}) = \phi(-\tilde{a}) = [\zeta \mapsto \zeta^{-1}]$ Makes sense that this is complex conjugation

~~[illegible]~~ § Norms of ideles

Last time we saw $Nm_{L/K} : I_L \to I_K$

extend this to $Nm_{L/K} : \mathbb{I}_L \to \mathbb{I}_K$

$$\prod L_w^\times \ni (a_w)_w \longmapsto \left(\prod_{w|v} Nm_{L_w/K_v} a_w\right)_v \in \prod K_v^\times$$

$$\begin{tikzcd} \mathcal{C}_L \arrow[r] \arrow[d, "Nm_{L/K}"'] & C_L \arrow[d, "Nm_{L/K}"] \\ \mathcal{C}_K \arrow[r] & C_K \end{tikzcd}$$

§ Main thms in terms of idèles

Recall Decomposition sbgp (of $w \in S_{p(L)}$; w prime of L lying over v): $D(w) = \{\sigma \in Gal(L/k) \mid \sigma w = w\}$

$$D(w) \cong Gal(L_w/k_v)$$

By local class field theory, local Artin map $\phi_v : K_v^\times \to D(w) \subset G$ independent of $w|v$ (essentially because Galois)

Prop 5.2 There is unique cts hom $\phi_K : \mathbb{I} \to Gal(k^{ab}/k)$ (Artin) s.t. $\forall L \subset K^{ab}$ fin ext $w|v$ w of L

$$\begin{tikzcd} K_v^\times \arrow[r, "\phi_v"] \arrow[d] & Gal(L_w/K_v) \arrow[d] \\ \mathbb{I}_K \arrow[r, "\tilde{a} \mapsto \phi_K(\tilde{a})|_L"] & Gal(L/k) \end{tikzcd}$$

"Pf" For fixed L, define $\phi_{L/K}(\tilde{a}) = \prod_v \phi_v(a_v)$ works.

By properties of local Artin maps, if $L' \supset L$, $\phi_{L'/K}|_L = \phi_{L/K}$.
Thus it extends to unique $\phi : \mathbb{I}_K \to Gal(k^{ab}/k)$
s.t. $\phi|_L = \phi_{L/K}$ skip the continuity

Thm (Artin Reciprocity): $\phi_K : \mathbb{I}_K \to Gal(k^{ab}/k)$ has properties:

(a) $\phi_k(k^\times) = 1$
(b) $\forall\, k \subset L$ fin ab ext $\Rightarrow$ $\cancel{\phi_{L/K}\ \mathbb{I}_K/(k^\times N_m(\mathbb{I}_L))}$

$$\phi_{L/K} : \mathbb{I}_K/(k^\times \cdot N_m(\mathbb{I}_L)) \xrightarrow{\sim} Gal(L/k)$$
$$\| $$
$$C_K/N_m(C_L)$$

E.g. $L = k[\sqrt[N]{a}]$, $\mu_N \subset k$ (a) says $\prod \phi_v(b) = 1$
$\Rightarrow$ Hilbert symbol prod formula $\prod_v (a,b)_v = 1$

Thm (Takagi existence) For every open $N \subset C_K$

$\exists!$ abelian extension L/K s.t. $Nm_{L/K} C_L = N$

called class field of K belonging to N

Cor bijection $\{$finite abelian extensions of $K\} \overset{1:1}{\longleftrightarrow} \{$open finite index subgps of $C_K\}$

Extra: K number field $\Rightarrow \phi_K$ surjective

K function field $\Rightarrow$ injective

$$C_K / Nm(C_L) \xrightarrow{\sim} Gal(L/K)$$

in limit $\quad Gal(K^{ab}/K) \cong \varprojlim_{\substack{N \text{ open} \\ \text{fin. index}}} C_K/N$