

$$G^{\text{ab}} = H_1^{-2}(G, \mathbb{Z}) \cong H_1^0(G, L^\times) = \mathbb{R}^\times / N_{\mathbb{R}}(L^\times) \text{ (Cohomological approach)}$$

CFT Preparation

Last time Global CFT in terms of ideals, earlier Local CFT

Thm (Artin Reciprocity): L/K fin ab ext S primes ram in L

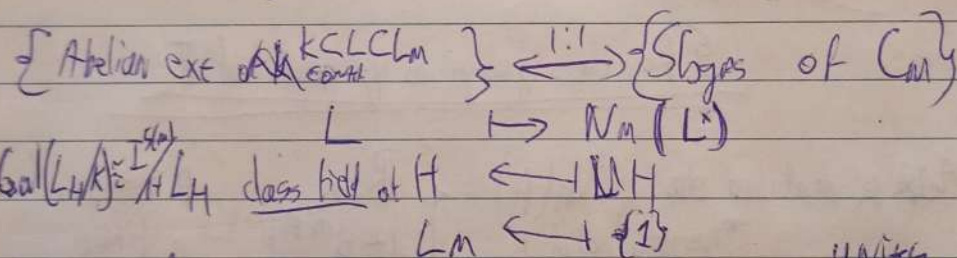
Then Artin map admits a modulus \bar{m} with $S(\bar{m}) \not\subseteq S$

$$I_K^{S(\bar{m})} / i(K_{\bar{m},1}) \cdot N_{L/K}(I_L^{S(\bar{m})}) \xrightarrow{\sim} \text{Gal}(L/K)$$

Thm (Takagi existence): Every congruence $i(K_{m,1}) \subset H \subset I_K^m$

is of the form $H = i(K_{m,1}) \cdot N_{L/K}(I_L^m)$ for some $L \subset K$
 \Rightarrow class field
finite abelian extension of primes dividing m

For $H = i(K_{m,1})$, $C_m \xrightarrow{\sim} \text{Gal}(L_m/K)$



Today Main thms in terms of idèles, unifies ~~connects~~ global CFT with LCF
 Reference V.4 & V.5 Milne's notes

§ Idèles and basic properties

$$C_m \xrightarrow{\sim} \text{Gal}(L_m/K)$$

$$\varprojlim_m C_m \xrightarrow{\sim} \text{Gal}(\bar{K}/K) \xrightarrow{\sim} \varprojlim_m \text{Gal}(L_m/K) = \text{Gal}(K^{\text{ab}}/K)$$

Replace by more natural topological gp: idèle class group C

Idea is to take all K_v 's together but with locally compact topology

Def Group of Idèles $\mathbb{I}_K = \prod_v K_v^\times = \left\{ (a_v) \in \prod_v K_v^\times \mid a_v \in \mathcal{O}_K \text{ almost all } v \right\}$

\mathbb{I}_K is not a topological group with induced adelic topology

Topology on \mathbb{I}_K : basis are sets $\prod V_v$

$V_v \subset K_v^x$ open
 $V_v = O_v$ almost all v

Recall Factor Surjective hom $\prod_K \rightarrow \mathbb{I}_K$
 $(a_v) \mapsto \prod P_v^{\text{ord}_v(a_v)}$
 \rightarrow thickening/extension of ideals
 $\forall v$ \prod_K also distinguishes between units in v finite or v infinite

i.e. between a and $-a$

~~Factor map~~

Def idèle class group $\mathbb{C}_K = \mathbb{I}_K / K^x$
 K^x is embedded through $a \mapsto (a, a, \dots)$
 \uparrow to \mathbb{I}

~~Def content of \mathbb{C}_K~~
 ~~$\mathbb{I} \cong \prod_v \mathbb{R}_{>0}$~~
 ~~$a \mapsto \prod |a_v|_v$~~

§ Ray class groups as quotients of \mathbb{I}

~~Def~~ modulus m $W_m(p) = \begin{cases} \mathbb{R}_{>0} & p \text{ real} \\ 1 + \mathfrak{p}^{m(p)} & p \text{ finite} \end{cases}$
 \rightarrow neighborhood of 1

Def $\mathbb{I}_m = \left(\prod_{p|m} K_p^x \times \prod_{p|m} W_m(p) \right) \cap \mathbb{I}$
 $a_p \in K_p^x \quad \forall p$
 $a_p \in O_p^x$ almost all p
 $a_p \in W_m(p) \quad p|m$

\mathbb{I}_m
 $W_m = \prod_{\substack{p|m \\ p \text{ int}}} K_p^x \times \prod_{p|m} W_m(p) \times \prod_{\substack{p|m \\ p \text{ fin}}} U_p$
 $a_p \in K_p^x \quad \text{int } p$
 $a_p \in O_p^x \quad \text{fin } p$
 $a_p \in W_m(p) \quad p|m$

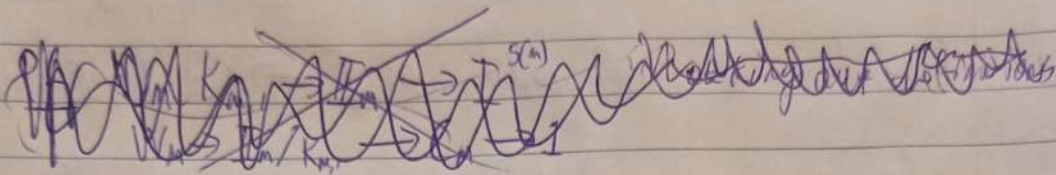
$K_{m,1} = K^x \cap \prod_{p|m} W_m(p)$ (int inside $\prod_{p|m} K_p^x$)

$\mathbb{I} = K^x \cap \mathbb{I}_m$ (inside \mathbb{I})

Prop 4.6 m modulus of K .

(a) $\mathbb{I}_m \rightarrow \mathbb{I}^{S(m)}$ gives isom $\mathbb{I}_m/K_{m,1} \cong W_m \cong C_m$

(b) $\mathbb{I}_m \hookrightarrow \mathbb{I}$ induces isom $\mathbb{I}_m/K_{m,1} \cong \mathbb{I}/K^x$



§ Characters of Galois extensions and norms

Let

G finite abelian gp. $\psi: \mathbb{I}^S \rightarrow G$ admits a modulus m

locally

if m has support in S and $\psi(i(K_{m,i})) = 1$

$\Rightarrow \psi$ factors through ray class group

Artin showed $\mathbb{I}^S \rightarrow \text{Gal}(L/K)$ admits a modulus, translate this to idèles

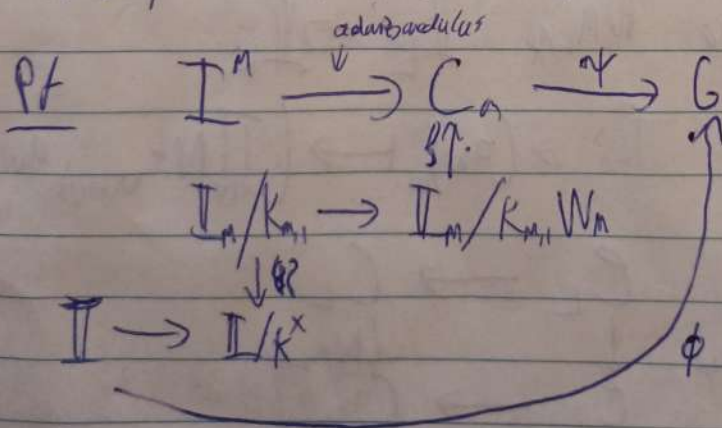
Prop 4.7 If $\psi: \mathbb{I}^S \rightarrow G$ admits a modulus, then $\exists! \phi: \mathbb{I} \rightarrow G$ of support S .

(a) ϕ is continuous

(b) $\phi(K^x) = 1$

(c) $\phi(a) = \psi(\text{id}(a)) \forall a \in \mathbb{I}^S := \{a \mid a_v = 1 \forall v \in S\}$

Every continuous $\phi: \mathbb{I} \rightarrow G$ arises from a ψ .



In practice, characterize $\mathbb{I} \times \phi$ by (a), (b), (c) instead of diagram

$$m = p \cdot \infty, S = \{p, \infty\}$$

Example $K = \mathbb{Q}, L = \mathbb{Q}[\sqrt{p}] \xrightarrow{\psi} \text{Artin map}$

$$I^S \longrightarrow C_M = (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \text{Gal}(L/\mathbb{Q})$$

$$(R/S) \longmapsto R S^{-1}, M \longmapsto [S \mapsto S^m]$$

Now determine corresponding $\phi: \mathbb{I} \longrightarrow \text{Gal}(L/\mathbb{Q})$ completely

$$(a_1, a_2, \dots, a_p, \dots, a_\infty) \in \mathbb{I}_{\mathbb{Q}} \quad \text{if } a_\infty = a_p = 1 \quad \text{then } \phi(a) = \psi(\text{id}(a))$$

$$(c) \phi(a) = \phi(\text{id}(a)) = [S \mapsto S^m] \quad m = \prod e_i^{ord(a_i)}$$

$$\tilde{p} = (1, \dots, \frac{1}{p}, 1, \dots) \quad \phi(\tilde{p}) = \phi(\tilde{p}/p) \phi(p) = 1$$

place p \neq not hard to show using continuity: $\phi(1)$ is same operation as 1.

$$\tilde{a} = (1, \dots, 1, u, 1, \dots) \quad \text{will give } \phi(\tilde{a})(S): S \mapsto S^c$$

some function $u \in \mathbb{Z}_p^\times$ will give $a_p = a_\infty = 1$

$$-\tilde{a} = (-1, 1, \dots, \frac{1}{p}, \dots) \quad -\tilde{a} = (1, -1, \dots, -\frac{1}{p}, \dots) \quad \text{above}$$

$$\phi(-\tilde{a}) = \phi(\tilde{a})^{-1} = [S \mapsto S^{-1}] \quad \text{Makes sense that this is complex conjugation}$$

~~Notes of idles~~ \S Notes of idles

Last time we saw $NM_{L/K}: \mathbb{I}_L \rightarrow \mathbb{I}_K$

extend this to $NM_{L/K}: \mathbb{I}_L \rightarrow \mathbb{I}_K$

$$\mathbb{I}_L^\times \ni (a_w)_w \longmapsto \left(\prod_{w|v} NM_{w/K} a_w \right)_v \in \mathbb{I}_K^\times$$

$$\begin{array}{ccc} \mathbb{C}_L & \longrightarrow & \mathbb{C}_L \\ \downarrow NM_{L/K} & & \downarrow NM_{L/K} \\ \mathbb{C}_K & \longrightarrow & \mathbb{C}_K \end{array}$$

§ Main thms in terms of idèles

Recall Decomposition stage: $D(w) = \{ \sigma \in \text{Gal}(L/K) \mid \sigma w = w \}$
w prime of K lying over v
of w in L

$$D(w) \cong \text{Gal}(L_w/K_v)$$

By local class field theory, local Artin map $\phi_v: K_v^\times \rightarrow D(w) \subset G$
 independent of $w|v$ (essentially because Galois)

Prop 6.2 There \exists unique cts hom $\prod \phi_k: \mathbb{I} \rightarrow \text{Gal}(K^{ab}/K)$
 s.t. $\forall L \subset K^{ab}$ fin ext $w|v$ prime of K $w \in \text{Gal}(L/K)$

$$\begin{array}{ccc} K_v^\times & \xrightarrow{\phi_v} & \text{Gal}(L_w/K_v) \\ \downarrow & & \downarrow \\ \mathbb{I}_K & \xrightarrow{\tilde{\alpha} \mapsto \phi_k(\tilde{\alpha})|_L} & \text{Gal}(L/K) \end{array}$$

Pf For fixed L , let $\phi_{L,K}(\tilde{\alpha}) = \prod \phi_v(\alpha_v)$ works

By properties of local Artin maps, if $L' \supset L$, $\phi_{L'/K}|_L = \phi_{L/K}$
 Thus it extends to unique $\phi: \mathbb{I}_K \rightarrow \text{Gal}(K^{ab}/K)$
 s.t. $\phi|_L = \phi_{L/K}$ Skip the continuity

Thm (Artin Reciprocity): $\phi_k: \mathbb{I}_K \rightarrow \text{Gal}(K^{ab}/K)$ has props:

(a) $\phi_k(K^\times) = 1$

(b) $\Delta \backslash \mathbb{I}_K \backslash L$ fin ab ext $\Rightarrow \phi_{L/K} \backslash \mathbb{I}_K / (K^\times \cdot \text{Nm}(\mathbb{I}_L))$

$$\begin{aligned} \phi_{L/K}: \mathbb{I}_K / (K^\times \cdot \text{Nm}(\mathbb{I}_L)) &\xrightarrow{\cong} \text{Gal}(L/K) \\ &\cong \mathbb{C}_K / \text{Nm}(\mathbb{C}_L) \end{aligned}$$

E.g. $L = K[\sqrt[n]{a}]$, $a \mu_n \subset K$ (a) says $\prod \phi_v(b) = 1$
 \Rightarrow Hilbert-Symbol prod formula $\prod_v (a, b)_v = 1$

P. Thm (Takagi existence) For every open $U \subset \mathbb{A}^1 \subset \mathbb{C}_k$

$\exists!$ abelian extension L/K s.t. $N_{L/K} \mathcal{O}_L = \mathcal{O}_U$
called class field of K belonging to U

(OR) bijess $\{\text{finite abelian extensions of } K\} \xleftrightarrow{1:1} \{\text{open finite index subsets of } \mathbb{C}_k\}$

Exerc: K number field $\Rightarrow \phi_K$ surjective
 K function field \Rightarrow injective

$$\mathbb{C}_k / N(\mathcal{O}_L) \xrightarrow{\sim} \text{Gal}(L/K)$$

$$\text{in limit} \quad \text{Gal}(K^{\text{ab}}/K) \cong \varinjlim_{\text{finite index}} \mathbb{C}_k / N$$