

Q: MOTIVATION

Last time: L/K finite Galois extension of local fields

$$0 \rightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z} \rightarrow H^2(K^{nr}/K) \xrightarrow{\text{Res}} H^2(L^{nr}/L)$$

$$\begin{matrix} G = \text{Gal}(L/K) \\ n = [L : K] \end{matrix}$$

$$H^0(L/K) \cong H^0(\bar{G}, L^\times)$$

Suppose we know that $\text{inv}_{L/K} : \frac{1}{n}\mathbb{Z}/\mathbb{Z} \rightarrow H^2(L/K)$ is an isomorphism.

$\text{inv}_{L/K}([u_{L/K}]) = [\frac{u}{n}]$.

Then by Tate's theorem $H_T^r(L, \mathbb{Z}) \rightarrow H_T^{r+2}(L^\times, L^\times)$ is an isomorphism.

$$x \mapsto x \cup_{L/K}$$

$r=2$: $L^{\text{ab}} \xrightarrow{\sim} K^\times / Nm(L^\times)$: write $\phi_{L/K}$ for the inverse.

Can repackage into a single map $\phi_K : K^\times \rightarrow \text{Gal}(K^{\text{ab}}/K)$

$$\boxed{LCFT \leftrightarrow H^2(L/K) \xrightarrow{\text{inv}} \frac{1}{n}\mathbb{Z}/\mathbb{Z}}$$

Brauer groups: $\{ \text{as } K\text{-algebras} \} / \text{similarity}$

Theorem: $\text{Br}(K) \cong H^2(K^{\text{sep}}/K)$

Remark: There is a fundamental SES -

2.14(e) $\text{defn} \Rightarrow 0 \rightarrow \text{Br}(K) \xrightarrow{\text{inv}} \bigoplus_i \text{Br}(k_v) \rightarrow \mathbb{Z}/2 \rightarrow 0$

\Rightarrow Hasse principle. quadratic form has a sm in $k \iff$ sm in k_v at places.

1: SIMPLE ALGEBRAS AND SEMISIMPLE MODULES

Definitions

A k-algebra is a ring containing k in its centre which is finite-dimensional as a k -vector space.

From a k -algebra $A \rightarrow A^{opp}$: $\beta \cdot \alpha = \alpha \beta$

A k -algebra is simple if it contains no proper two-sided ideals.

A k -algebra is a division algebra if $x \neq 0 \Rightarrow x$ has an inverse.

Examples: • $H = \text{M}_2(k, i, j, k) / \{z^2 = jk = -1\}$.

Ex 1.11 • Division algebra $\rightarrow \text{M}_n(D)$ is a simple k -alg.

An A -module is a finitely-generated left A -module V .

V is simple if $V \neq 0$ and contains no proper A -submodules.

Theorem: If A is a simple k -algebra, then any two simple A -modules are isomorphic. (Th 1.19(c))
Tensor products

Usual \otimes of k -vector spaces, with $(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb')$.

Identify $k \leftrightarrow k(1 \otimes 1)$. Then $A \otimes_k B$ is a k -algebra.

Ex 2.1 { For a k -algebra A , $A \otimes_k \text{M}_n(k) \cong \text{M}_n(A)$.

Even more, $A \otimes_k \text{M}_n(A') \cong \text{M}_n(A \otimes_k A')$.

Centralisers

Let A be a k -subalgebra of a k -algebra B .

The centraliser of A in B is $C_B(A) = \{b \in B \mid ab = ba \forall a \in A\}$.

A k -algebra A is central if $C_A(A) = k$.

Examples

$C_{M_n(k)} M_n(k) = k I_n$.
• $C_{M_n(k)}$ is central simple k -alg.
• $M_n(k)$ is central simple k -alg.

2.3 Proposition: Let A, A' be k -algebras with B, B' k -subalgebras.

Then $C_{B \otimes B'}(A \otimes A') = C_B(A) \otimes C_{B'}(A')$.

(3)

Lemma (Schur): Let V be a simple A -module. Then $\text{End}_A(V)$ is a division algebra.

1.46 Proof: $\gamma \in \text{End}_A(V) \quad \gamma: V \rightarrow V$
 $\gamma \neq 0. \quad \ker(\gamma) = 0.$
 $\therefore \gamma$ an isomorphism.

Theorem (Wedderburn): Let A be a simple k -algebra. Then there exists $n \geq 1$ and a division k -algebra D such that $A \cong M_n(D)$. Also, n and D are uniquely determined by A .

Proof:

Supplement to BigW: Say that two k -algebras A and B are similar if $\exists m, n$ such that

$$A \underset{k}{\otimes} M_n(k) \cong B \underset{k}{\otimes} M_m(k).$$

If A_1 and A_2 are similar, simple k -algebras, then $D_1 \cong D_2$.
 Proof: $\left. \begin{array}{l} A_1 \cong M_{n_1}(D_1) \\ A_2 \cong M_{n_2}(D_2) \end{array} \right\} \begin{array}{l} D_1 \otimes M_{n_1 n_2}(k) \cong D_2 \otimes M_{n_1 n_2}(k) \\ M_{n_1 n_2}(D_1) \cong M_{n_1 n_2}(D_2) \end{array}$
 $\left. \begin{array}{l} M_{n_1}(D_1) \otimes M_{n_2}(k) \cong \underbrace{M_{n_2}(D_2)}_{D_2 \otimes M_{n_2}(k)} \otimes M_{n_1}(k) \\ D_1 \otimes M_{n_1}(k) \end{array} \right\} D_1 \cong D_2.$

Proposition: Let A and B be simple k -algebras, and suppose (at least) one of A or B is central.

2.6 Then $A \underset{k}{\otimes} B$ is simple. 2.7

Proof (assuming the result for at least one of A or B a division algebra):
 $A \cong M_n(D) \quad A \underset{k}{\otimes} B \cong M_n(D \underset{k}{\otimes} B) \cong M_n(M_n(D')) \cong M_{nn}(D')$.

Corollary: $(\text{central simple})_k^{\otimes} (\text{central simple})$ is central simple.

$$\text{Proof: } C(A \otimes_k B) = C(A) \otimes_k C(B) = k \otimes_k k = k.$$

Corollary: For a central simple k -algebra A , $A \otimes_k A^{opp} \cong \text{End}_k(A) \cong M_{[A:k]}(k)$.

2.9 Proof: Let $V = "A \text{ regarded as a } k\text{-vector space}"$.

- V is a left A -module (left mult)
 - V is a left A^{opp} -module (right mult)
- $$A \otimes_k A^{opp} \xrightarrow{\varphi} \text{End}_k(V)$$
- $$a \otimes a' \mapsto L(v) = ava'$$

$$\begin{aligned} \text{By Prop. 2.6, } \ker \varphi &= 0 \text{ or all of } \underbrace{x \cdot 1_C}_{x \in A} \\ &\quad \boxed{[A \otimes_k A^{opp}; k] = [A; k]^2 = \dim_k \text{End}_k V} \\ &\quad \ker \varphi = 0. \end{aligned}$$

Theorem (Noether-Skolem): Let A, B be k -algebras, $f, g: A \rightarrow B$ homomorphisms.

2.10 If A is simple, B central simple, then $\exists b \in B$ invertible with:

$$\boxed{f(a) = b \cdot g(a) \cdot b^{-1} \quad \forall a \in A}$$

Corollary: Let A be a CSA over k :

2.11 B_1, B_2 simple subalgebras. For any isomorphism $f: B_1 \rightarrow B_2$, $\exists a \in A$ invertible with $f(b) = aba^{-1} \quad \forall b \in B_1$.

$$\begin{aligned} \text{Proof: } B_1 &\xrightarrow{f} B_2 \hookrightarrow A \\ B_1 &\xrightarrow{\text{id}} B_1 \hookrightarrow A \end{aligned}$$

2: DEFINITION OF THE BRAUER GROUP

Definition: Let k be a field. $\text{Br}(k) = \{\text{central simple } k\text{-algebras}\} / \text{similarity}$
 $A \sim B \Leftrightarrow A \otimes_{\mathbb{K}} M_n(k) \cong B \otimes_{\mathbb{K}} M_n(k)$.

$$\text{Set } [A][B] = [A \otimes_{\mathbb{K}} B].$$

$\rightarrow \text{Br}(k)$ is an abelian group.
 $\text{id} = [M_n(k)] \text{ any } n > 1.$

Recall supplement to Wedderburn's theorem:

$A \sim B$, D and $B \sim D'$. Then $D \sim D'$.

So $\text{Br}(k) = \{\text{central division } k\text{-algebras}\} / \text{isomorph.}$

Note: $C_S \otimes C_S = C_S$ $D_{IV} \otimes D_{IV} = D_{IV}.$

Examples: **2.14**

$$\textcircled{1} \quad \text{Br}(\mathbb{R}) = \{[\mathbb{R}] , [\mathbb{H}] \} \quad (\text{Fröbenius' theorem})$$

$$\text{Br}(\mathbb{C}) \cong H^2(\mathbb{C}/\mathbb{R}) \cong \text{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}_2.$$

$$\text{Let } D = k[x]/(x^2 - 1).$$

$$k[x]/(x) = k(x).$$

$$k[x]/(x^2 - 1) = k.$$

Wedderburn's little theorem.

$\textcircled{3}$ k a finite field $\rightarrow \text{Br}(k) = 0$

Suppose we know that all finite division algebras are commutative.

Pick D . D finite. $C(D) = D$

$$K.$$

$\textcircled{4}$ k a non-arch. local field $\Rightarrow \text{Br}(k) \cong \mathbb{Q}/\mathbb{Z}$ canonically

⑥ Think geometrically: want to define a "relative" Brauer group $\text{Br}(L/K)$ so that $\text{Br}(k) \cong \text{Br}(\bar{k}/k)$.

Something like: $\text{Br}(L/K) = \{ \text{elements of } \text{Br}(k) \text{ that are trivial as elements of } \overbrace{\text{Br}(L)}^? \}$

$$\text{Br}(k) \xrightarrow{?} \text{Br}(L)$$

Proposition: Let L be a field containing K , A a CS k -alg.

2.15 Then $A \otimes_K L$ is a CS L -algebra.

$$\text{Proof: } C(A \otimes_L L) = L \otimes_K L = L.$$

$$A \cong M_n(D).$$

$$A \otimes_K L = M_n(D) \otimes_K L = M_n(k) \otimes_K D \otimes_K L$$

$$= M_n(L) \otimes_K (D \otimes_K L)$$

$$S/L.$$

- * L/K need not be finite, so
- * isn't a k -algebra. So
- * $CS \otimes S$ doesn't apply.
- (directly)

- * Remark: want this for
- * $k \otimes k$ univ.
- * Remark: all happening inside
- a field $k \otimes k$.

2.16 Then have a hom: $\text{Br}(k) \longrightarrow \text{Br}(L)$

Set $\text{Br}(L/K) = \text{the kernel: the } CS/k \text{ s split by } L.$

Proposition: For any field K , $\text{Br}(K) = \bigcup \text{Br}(K \otimes L/K)$, where the union is over L s.t. L/K finite.

2.17

Proof: A a K -CSA, wts $\exists L/K$ s.t. $A \otimes_L \cong M_n(L)$

$$1) A \otimes_K \cong M_n(D) \otimes_K \cong M_n(D \otimes_K k) \cong M_n(k).$$

$$2) \nearrow \text{This has basis } \{e_{ij}\}_{i,j=1}^n \quad e_{ij} e_{lm} = \delta_{ij} \delta_{lm}.$$

$$\bigcup_{L/K \text{ fin}} A \otimes_L \text{ so } \dim A \otimes_L.$$

3: BRAUER GROUPS AND COHOMOLOGY

$$H^2(L/K) := H^2(\text{Gal}(L/K), L^\times). \quad \boxed{\text{WTS} \quad Br(L/K) \cong H^2(L/K) \text{ canonically.}} \quad 3.11 - 3.15$$

L/K finite, Galois. $G = \text{Gal}(L/K)$

Technical condition...

Set $\mathcal{A}(L/K) = \{ \text{CSAS}_L^A / K, \text{ containing } L, \text{ with } [A : K] = [L : K]^2 \} / \text{iso}.$

PART ONE WTS $\mathcal{A}(L/K) \longrightarrow Br(L/K)$ is a bijection

$$[A] \mapsto [A].$$

Proof of injectivity: ~~Rek-Area~~ If $[A_1] = [A_2]$ in Br.

$$\begin{aligned} A_1 &\cong M_{n_1}(D) \\ A_2 &\cong M_{n_2}(D) \end{aligned} \quad \begin{aligned} [A_1 : K] &= [L : K]^2 = [A_2 : K] \\ n_1 &= n_2. \end{aligned}$$

Proof of surjectivity: Nasty.

PART TWO WTS there is a bijection $\gamma: \mathcal{A}(L/K) \longrightarrow H^2(L/K)$.

1) Defining γ :

Pick A rep a class in $\mathcal{A}(L/K)$.
 For $\sigma \in G$, by Cor to N-S $\exists e_\sigma \in A$ s.t. $\begin{cases} \sigma a = e_\sigma a e_\sigma^{-1} \\ e_\sigma a = \sigma a \cdot e_\sigma \end{cases}$ (*).

$e_\sigma e_\tau$ has (*) for $\sigma \mapsto \sigma \tau$

$$e_\sigma e_\tau = \varphi(\sigma, \tau) \cdot e_{\sigma \tau} \quad \varphi: (\sigma, \tau) \mapsto \sigma \tau$$

Check: φ is a 2-cocycle, $\varphi_{11} = \text{id}_{\text{new}}$ / boundary.
Check: If choose different e_σ , $\varphi_{11} = \text{id}_{\text{new}}$.

2) Well-definedness:

$$\begin{aligned} [A] &= [A'] \text{ in } \mathcal{A}. \\ \text{By N-S, } f: A &\longrightarrow A' \text{ s.t. } f|_L = \text{id} \text{ on } L. \\ f(e_\sigma a) &= \sigma a \cdot e_\sigma \text{ has L.} \end{aligned}$$

$$\begin{aligned} f(e_\sigma a) &= \sigma a \cdot e_\sigma \text{ has L.} \\ f(e_\sigma a) &= \sigma a \cdot f(e_\sigma). \end{aligned}$$

⑧

3) γ is injective:

Lemma: If A represents a class in ML/K , and for $e \in G$ I have $e_0 \in A$ satisfying $\sigma_a = e_0 \sigma_{e^{-1}} \forall a \in L$, then $\{e_0\}_{e \in G}$ is a basis for A as a left vector space over L .

4) γ is surjective:

PART THREE Simplifying WTS $\gamma(\varphi * \psi') = A(\varphi) \otimes_K A(\psi')$.

3.16

Corollary Let k^{sep} be a separable algebraic closure of k . Then $\text{Br}(k) \xrightarrow{\sim} H^2(k^{sep}/k)$.

(canonically)

Proof:

$$\begin{array}{ccccc}
& E & \xrightarrow{\varphi} & H^2(L/k) & \xrightarrow{hf} H^2(E/k) \\
& \downarrow & & \downarrow & \downarrow \\
L & \xrightarrow{\text{finite}} & Br(L/k) & \xrightarrow{\text{finite}} & Br(E/k) \\
\downarrow & & \downarrow & & \downarrow \\
K & Br(k) = \bigcup_{L/k} Br(L/k). & H^2(k^{sep}/k) = \bigcup_{L/k} H^2(L/k).
\end{array}$$

4: BRAUER GROUPS OF SPECIAL FIELDS

① Ch III, 1.4 : For k finite, L/k finite, $H^2(L/k) = 0$, so $\text{Br}(L/k) = 0$.

Or, Wedderburn's Little Theorem: Every finite division algebra is commutative.

3.4.1

Proof:

* No finite group may be written as a union of conjugates of proper subgroups.

② We know that $\text{Br}(k) = \{[H]\}, [H] \neq 1$ now:

Explicitly: there is a nontrivial 2-wedge $\varphi: G \times_G \rightarrow \mathbb{C}^\times : \varphi([\rho], [\sigma]) = \begin{cases} 1 & \rho = \sigma \\ -1 & \text{otherwise} \end{cases}$

Then $A(\varphi) \simeq H$.

Important: ③ $\text{Br}(K) \cong \mathbb{R}/\mathbb{Z}$ for K nonarch. local.

Sketch: ① $\text{Br}(K) = \text{Br}(K^{\text{ur}}/K)$

2) $\text{inv}_K : \text{Br}(K) \rightarrow \mathbb{R}/\mathbb{Z}$ by:

a) Pick $D \in \text{Br}(K)$

b)

* ord is an extension of the additive valuation on K , to D .

c) $d) \text{inv}_K(D) := \text{ord}(\alpha) \bmod \mathbb{Z}.$

3) Show inv_K is a bijection: Actually show that inv_K is an iso: $\text{Br}(K^{\text{ur}}/K) \rightarrow \mathbb{R}/\mathbb{Z}$,
(and isomorphism...) then use 1).