Logic and Graphons

Algorithms, Logic and Structure

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Our friend the graphon

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More importantly for us, the graphon space is actually a subspace of an ultraproduct of a sequence of finite graphs, as discovered by Elek and Szegedy in 2007.

Ultrafilters

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Theorem (Łoś 1955) For any first-order formula $\varphi(x)$ and $\bar{a} \in \prod_{\alpha<\kappa} \mathfrak{A}_\alpha/\mathcal{U}$ we have

$$\prod_{\alpha<\kappa} \mathfrak{A}_\alpha/\mathcal{U} \models \varphi[\bar{a}] \text{ iff } \{\alpha < \kappa : \mathfrak{A}_\alpha \models \varphi[a_\alpha]\} \in \mathcal{U}.$$

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Let F be a field (finite), $M \in \omega$, we consider subsets of F definable by a formula of complexity M.

Theorem (Tao 2013) (Algebraic Regularity Lemma) For every M there is $C = C_M > 0$ such that for any finite field F of **characteristic** $\geq C$, $\emptyset \neq V, W \subseteq F, E \subseteq V \times W$ all definable of complexity $\leq M$, there exist partitions of V into $a \leq C$ and W into $b \leq C$ pieces $V_i (i < a), W_j (j < b)$:

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- such that $|V_i| \geq |V|/C, |W_j| \geq |W|/C$ and V_i, W_j are definable of complexity $\leq C$ and
- if $A \subseteq V_i, B \subseteq W_j$ then
$$\left||E \cap (A \times B)| - \frac{|E \cap (V_i \times V_j)|}{|V_i||W_j|}\right| \leq C|F|^{-1/4}|V_i||W_j|.$$

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Theorem Let Γ be a parameter-free definable bipartite graph. The set of accumulation points of the family of finite graphs

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The same ideas apply to schemes more generally.

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Chernikov and Starchenko (to appear) give an elegant short proof of the above theorem using dimensions in ultraproducts developed by Hrushovski. Ivan and I are looking to get a proof using graphons and to extend it to NIP.

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This measure is not separable so the algebra is not isomorphic to that of $[0,1]$ with Lebesgue measure, rather to that of $[0,1]^{\lambda}$ with the product measure, for some λ (Maharam's theorem).

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Theorem (Kunen 1972) For every κ there are 2^{2^κ} many countably incomplete 'good' ultrafilters over κ.

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These ultrafilters do not give rise to the Loeb meaasure, but we can develop the theory of definability and dimensions similar to what was done in geometric stablity theory fo the countable case.

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