TUTORIAL #1: BASIC CONCEPTS AND EXAMPLES

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(1) A sequence of graphons $W_n \colon [0,1]^2 \to [0,1]$ is *weak* convergent* to W if $\int_S W_n \to \int_S W$ for every measurable $S \subseteq [0,1]^2$. Does weak* convergence imply cut norm convergence? How about vice versa? Find a proof or counterexample for each direction. How do weak* and cut norm convergence compare with L^1 convergence?

(2) Prove or disprove: the best cut norm approximation of a graphon $W \colon [0,1]^2 \to [0,1]$ by a step function constant on $\mathcal{P}_i \times \mathcal{P}_j$ (where $\mathcal{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_n\}$ is a partition of $[0,1]$) is the average $W_{\mathcal{P}}$ of W with respect to $\mathcal{P}$.

(3) Let S and T be finite sets, and let $f \colon S \times T \to \mathbb{R}$ be any function. Prove that

$$\left(\sum_{\substack{s \in S \\ t \in T}} f(s,t) \right)^2 \le |S| \sum_{\substack{s \in S \\ t_1, t_2 \in T}} f(s,t_1) f(s,t_2).$$

(4) Let $(G_n)_{n \ge 0}$ be a sequence of graphs such that G_n has n vertices, and let $0 < p < 1$. Suppose the number of edges in G_n is $(1+o(1))pn^2/2$ as $n \to \infty$. Prove that the number of 4-cycles in G_n (i.e., quadruples (v_1, v_2, v_3, v_4) of vertices with v_i adjacent to v_{i+1}, where $v_5 = v_1$) is at least $p^4 n^4 (1+o(1))$.

(5) Let $(G_n)_{n \ge 0}$ be a sequence of graphs such that G_n has n vertices, and let $0 < p < 1$. Suppose the vertices of G_n have degrees $pn(1+o(1))$ as $n \to \infty$, and each pair of distinct vertices in G_n has $p^2 n(1+o(1))$ common neighbors. Prove that G_n converges to the constant graphon p under the cut norm.

(6) Let $0 < p < 1$, and consider an Erdős-Rényi graph $G_{n,p}$ with edge probability p. In other words, we take n vertices and flip an independent coin to decide whether there's an edge between each pair of distinct vertices, with probability p of an edge. Prove that with probability 1, these graphs converge to the constant graphon with value p on $[0,1]^2$ as $n \to \infty$.

(7) Let p be a prime congruent to 1 modulo 4, and define a graph on p vertices by connecting vertices i and j with an edge if $i - j$ is a nonzero square modulo p. Note that $p \equiv 1 \pmod 4$ ensures that the edge relation is symmetric; the resulting graph is called a *Paley graph*. Prove that as $p \to \infty$, the Paley graphs converge to the constant $1/2$ graphon. (No special background is needed beyond the fact that $\mathbb{Z}/p\mathbb{Z}$ is a finite field.)