Lecture 2 - From weak regularity to strong regularity

In this lecture, we will give a slightly different approach to regularity, starting from the weak regularity lemma of Frieze and Kannan.

Theorem 1 *For every $\epsilon > 0$, there exists a K such that, for every graph G, there is a partition $V(G) = X_1 \cup \dots X_k$ into $k \leq K$ parts such that for all $A, B \subseteq V(G)$,*

$$|e(A,B) - \sum_{1 \leq i,j \leq k} d(X_i, X_j)|A \cup X_i||B \cap X_j|| \leq \epsilon |V|^2.$$

This differs from the usual regularity lemma because it only gives global control over the density between two sets, while the ordinary regularity lemma gives local control inside the sets of the partition. The proof of this lemma, which we will omit, follows the same lines as the proof of the regularity lemma. However, one does not need to take a common refinement at every step. As a consequence, the bounds for the weak regularity lemma are surprisingly reasonable. We may take $K = 2^{O(\epsilon^{-2})}$.

Starting from the Frieze–Kannan regularity lemma, we may give an alternative proof of the ordinary regularity lemma. In fact, we get a somewhat stronger result. Before stating the result, we will introduce some notation. Given a partition $\mathcal{A} = X_1 \cup \dots \cup X_k$ of the vertex set of a graph on n vertices, we will write

$$q(\mathcal{A}) = \sum_{1 \leq i,j \leq k} \frac{|X_i||X_j|}{n^2} d(X_i, X_j)^2$$

for the mean square density of the partition.

Theorem 2 *For any $\eta, \epsilon > 0$ and every function $f : \mathbb{N} \to (0, 1]$, there exists T such that, for every graph G, there exists a partition of the vertex set $\mathcal{A} = X_1 \cup \dots \cup X_k$ and a refinement $\mathcal{B} = Y_1 \cup \dots \cup Y_\ell$ with $\ell \leq T$ such that*

(i) *$\mathcal{A}$ is a weak ϵ-regular partition;*

(ii) *$\mathcal{B}$ is a weak $f(k)$-regular partition;*

(iii) $q(\mathcal{B}) \leq q(\mathcal{A}) + \eta$.

Proof Without loss of generality, we may assume that $f(i) \leq \epsilon$ for all i. We apply Theorem 1 to G to find a weak ϵ-regular partition with k_1 parts. We denote this partition by $\mathcal{P}_1$. We then apply the theorem again to find a weak $f(k_1)$-regular partition with k_2 parts. If $q(\mathcal{P}_2) \leq q(\mathcal{P}_1) + \eta$, we will be done. We may therefore assume that this is not the case.

Suppose now that we have arrived at partitions $\mathcal{P}_{i-1}$ and $\mathcal{P}_i$ with k_{i-1} and k_i parts, respectively, such that $\mathcal{P}_{i-1}$ is weak ϵ-regular and $\mathcal{P}_i$ is $f(k_{i-1})$-regular. If $q(\mathcal{P}_i) \leq q(\mathcal{P}_{i-1}) + \eta$, we are done. We may therefore assume that $q(\mathcal{P}_i) > q(\mathcal{P}_{i-1}) + \eta$. However, since the mean square density is bounded above by 1, this can continue for at most $1/\eta$ steps. Therefore, we must eventually arrive at the required pair of partitions. □

In order to see why this implies the regularity lemma, we need a lemma showing that if $q(\mathcal{B}) \leq q(\mathcal{A}) + \eta$, then $\mathcal{A}$ and $\mathcal{B}$ look similar to one another.

Lemma 1 *Let $\mathcal{A} = X_1 \cup \cdots \cup X_k$ be a partition of the vertex set of a graph with n vertices and let $\mathcal{B} = X_{11} \cup \cdots \cup X_{1a_1} \cup \cdots \cup X_k \cup \cdots \cup X_{ka_k}$ be a refinement of this partition. If $q(\mathcal{B}) \leq q(\mathcal{A}) + \eta$, then*

$$\sum_{1 \leq i,j \leq k} \sum_{a,b} \{ \frac{|X_{ia}||X_{jb}|}{n^2} : |d(X_{ia}, X_{jb}) - d(X_i, X_j)| \geq \gamma \} \leq \frac{\eta}{\gamma^2}.$$

Proof Let $\Delta = \sum_{i,j} \sum_{a,b} \{ \frac{|X_{ia}||X_{jb}|}{n^2} : |d(X_{ia}, X_{jb}) - d(X_i, X_j)| \geq \gamma \}$. If we write $u(X_{ia}, X_{jb}) = d(X_{ia}, X_{jb}) - d(X_i, X_j)$, we see that

$$\begin{aligned} q(\mathcal{B}) &= \sum_{1 \leq i,j \leq k} \sum_{a,b} \frac{|X_{ia}||X_{jb}|}{n^2} d(X_{ia}, X_{jb})^2 = \sum_{1 \leq i,j \leq k} \sum_{a,b} \frac{|X_{ia}||X_{jb}|}{n^2} (d(X_i, X_j) + u(X_{ia}, X_{jb}))^2 \\ &= \sum_{1 \leq i,j \leq k} \sum_{a,b} \frac{|X_{ia}||X_{jb}|}{n^2} d(X_i, X_j)^2 + 2 \sum_{1 \leq i,j \leq k} \sum_{a,b} \frac{|X_{ia}||X_{jb}|}{n^2} d(X_i, X_j) u(X_{ia}, X_{jb}) \\ &+ \sum_{1 \leq i,j \leq k} \sum_{a,b} \frac{|X_{ia}||X_{jb}|}{n^2} u(X_{ia}, X_{jb})^2 \\ &\geq q(\mathcal{A}) + \Delta \gamma^2, \end{aligned}$$

where we used that $\sum_{a,b} \frac{|X_{ia}||X_{jb}|}{n^2} u(X_{ia}, X_{jb}) = 0$ and the definition of Δ. Comparing this inequality with the assumption that $q(\mathcal{B}) \leq q(\mathcal{A}) + \eta$ implies the result. □

With this preliminary, we may now give another proof of the regularity lemma.

Second proof of the regularity lemma: We apply Theorem 2 with ϵ, $f(t) = \epsilon^3/4t^2$ and $\eta = \epsilon^{10}/64$. This gives a partition $\mathcal{A} = X_1 \cup \cdots \cup X_k$ and a refinement $\mathcal{B} = X_{11} \cup \cdots \cup X_{1a_1} \cup \cdots \cup X_k \cup \cdots \cup X_{ka_k}$. In this proof, it will be slightly more convenient to assume that the partitions are equitable, so we will do so. We will now show that $\mathcal{A}$ is the required partition.

For every pair $1 \leq i, j \leq k$, let

$$A_{ij} = \{(a, b) : |d(X_{ia}, X_{jb}) - d(X_i, X_j)| \geq \epsilon^3/4\}.$$

We now let

$$I = \{(i, j) : \sum_{(a,b) \in A_{ij}} \frac{|X_{ia}||X_{jb}|}{|X_i||X_j|} \geq \epsilon^3/4\}.$$

We claim that $\sum_{(i,j) \in I} \frac{|X_i||X_j|}{n^2} \leq \epsilon$. To see this, note that

$$\sum_{i,j} \sum_{a,b} \{ \frac{|X_{ia}||X_{jb}|}{n^2} : |d(X_{ia}, X_{jb}) - d(X_i, X_j)| \geq \epsilon^3/4 \} \geq \frac{\epsilon^3}{4} \sum_{(i,j) \in I} \frac{|X_i||X_j|}{n^2}.$$

Since Lemma 1 shows that this sum is at most $\epsilon^4/4$, the claim follows.

Suppose now that $(i, j) \notin I$. We will show that the pair (X_i, X_j) is ϵ-regular. Suppose then that $A \subseteq X_i$ and $B \subseteq X_j$ with $|A| \geq \epsilon |X_i|$ and $|B| \geq \epsilon |X_j|$. Since the partition $\mathcal{B}$ is $f(k)$-regular, we have that

$$|e(A, B) - \sum_{a,b} d(X_{ia}, X_{jb}) |A \cup X_{ia}||B \cap X_{jb}|| \leq f(k) n^2.$$

Moreover, since $(i,j) \notin I$, we have

$$\begin{aligned}\sum_{a,b} |d(X_i, X_j) - d(X_{ia}, X_{jb})||A \cup X_{ia}||B \cap X_{jb}| &\leq \sum_{a,b} |d(X_i, X_j) - d(X_{ia}, X_{jb})||X_{ia}||X_{jb}| \\ &\leq \left(\frac{\epsilon^3}{4} + \frac{\epsilon^3}{4}\right)|X_i||X_j| = \frac{\epsilon^3}{2}|X_i||X_j|.\end{aligned}$$

Therefore, since $f(k)n^2 = \epsilon^3 n^2/4k^2 = \epsilon^3|X_i||X_j|/4$, we have

$$|e(A,B) - d(X_i, X_j)|A||B|| \leq f(k)n^2 + \frac{\epsilon^3}{2}|X_i||X_j| \leq \epsilon^3|X_i||X_j|.$$

Dividing through by $|A||B| \geq \epsilon^2|X_i||X_j|$ gives the result. □

In the same way that one iterates the weak regularity lemma to get the usual regularity lemma, one can iterate the usual regularity lemma to prove a strong regularity lemma. This lemma, due to Alon, Fischer, Krivelevich and Szegedy, plays a central role in the theory of property testing.

Theorem 3 *For any $\eta, \epsilon > 0$ and every function $f : \mathbb{N} \to (0,1]$, there exists T such that, for every graph G, there exists a partition of the vertex set $\mathcal{A} = X_1 \cup \cdots \cup X_k$ and a refinement $\mathcal{B} = Y_1 \cup \cdots \cup Y_\ell$ with $\ell \leq T$ such that*

(i) $\mathcal{A}$ is an ϵ-regular partition;

(ii) $\mathcal{B}$ is an $f(k)$-regular partition;

(iii) $q(\mathcal{B}) \leq q(\mathcal{A}) + \eta$.

The proof of this theorem gives a bound for T which is wowzer in a power of ϵ^{-1}, even for reasonable functions f. The wowzer function is obtained by iterating the tower function, that is, $W(1) = 2$ and $W(i) = T(W(i-1))$. As with the regularity lemma, there are graphs for which a bound of this type is necessary, as proved by the author and Fox and by Kalyanasundaram and Shapira.