Problem Set 1 - counting lemmas and the triangle removal lemma

1. Suppose that $d > \epsilon > 0$ and X and Y are disjoint sets such that (X, Y) is an ϵ-regular pair with density at least d. Prove that there are at least $(1-\epsilon)|X|$ vertices $x \in X$ such that $|N(x) \cap Y| \geq (d-\epsilon)|Y|$, where $N(x)$ is the neighbourhood of x.

2. Suppose that $d > 2\epsilon > 0$ and X, Y and Z are disjoint set such that (X, Y), (Y, Z) and (Z, X) are all ϵ-regular pairs with density at least d. Prove that there exists $\eta > 0$ such that there are at least $\eta|X||Y||Z|$ triangles with one vertex in each of X, Y and Z.

3. Use the regularity lemma and the exercise above to prove the triangle removal lemma: for any $\epsilon > 0$, there exists a $\delta > 0$ such that if G is a graph on n vertices with fewer than δn^3 triangles, it may be made triangle-free by removing at most ϵn^2 edges.

4. Use the triangle removal lemma to prove Roth's theorem: for every $\delta > 0$, there exists n_0 such that, for $n \geq n_0$, every subset A of $\{1, 2, \ldots, n\}$ of size δn contains a 3-term arithmetic progression. [*Hint: Consider the graph between three copies X, Y and Z of $[3n]$, where xy is an edge if $y - x \in A$, yz is an edge if $z - y \in A$ and zx is an edge if $z - x \in 2A$.*]

5. Use the triangle removal lemma to prove the induced matching lemma: any graph on n vertices which is the union of n induced matchings has $o(n^2)$ edges.

6. Give another proof of Roth's theorem using the induced matching lemma. [*Hint: Consider the bipartite graph between X and Y, where X is a copy of $[2n]$ and Y is a copy of $[3n]$, and look at the matchings formed by fixing x and joining $x + a$ to $x + 2a$ for each $a \in A$.*]