# Problem Set 3 - convexity, weak regularity and more

1. Show that for any convex function $q$ and any partition $\mathcal{A}$ of a graph with density $d$,
$$q(d) \le q(\mathcal{A}) \le dq(1) + (1-d)q(0).$$

2. Show that if $\mathcal{A}$ and $\mathcal{B}$ are partitions of a graph with $\mathcal{B}$ a refinement of $\mathcal{A}$ and $q$ is a convex function, then $q(\mathcal{B}) \ge q(\mathcal{A})$. [*In Fox's proof of the removal lemma, we used a defect version of this inequality.*]

3. Given $g : X \times Y \to [0,1]$, viewed as an edge-weighted bipartite graph between $X$ and $Y$, the cut norm of $g$ is
$$\|g\|_\square = \sup_{A \subseteq X, B \subseteq Y} |\mathbb{E}_{x \in X, y \in Y} g(x,y) 1_A(x) 1_B(y)|.$$
[*With this definition, the weak regularity lemma then says that for any $g$ there exists a graph $\tilde{g}$ with complexity bounded in terms of $\epsilon$ such that $\|g - \tilde{g}\|_\square \le \epsilon$.*]

   Show that if $\|g\|_\square \le \epsilon$, this implies that
$$|\mathbb{E}_{x \in X, y \in Y} g(x,y) a(x) b(y)| \le \epsilon$$
for any functions $a : X \to [0,1]$ and $b : Y \to [0,1]$.

4. Show that if $f, \tilde{f} : X \times Y \to [0,1]$, $g, \tilde{g} : Y \times Z \to [0,1]$ and $h, \tilde{h} : Z \times X \to [0,1]$ are functions such that $\|f - \tilde{f}\|_\square \le \epsilon$, $\|g - \tilde{g}\|_\square \le \epsilon$ and $\|h - \tilde{h}\|_\square \le \epsilon$, then
$$|\mathbb{E}_{x \in X, y \in Y, z \in Z} f(x,y)g(y,z)h(z,x) - \tilde{f}(x,y)\tilde{g}(y,z)\tilde{h}(z,x)| \le 3\epsilon.$$
[*This is the triangle counting lemma corresponding to the weak regularity lemma.*]

5. How should one define a hypergraph cut norm so that a corresponding counting lemma holds?

6. The half graph is the bipartite graph between sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ with edge set $\{x_i y_j : i \le j\}$. Show that any $\epsilon$-regular partition of the half graph must contain an irregular pair.