On the Structure of Selmer Groups

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Dedicated to John Coates.

1 Introduction

Our objective in this paper is to prove a rather broad generalization of some classical theorems in Iwasawa theory. We begin by recalling two of those old results. The first is a theorem of Iwasawa, which we state in terms of Galois cohomology. Suppose that K is a totally real number field and that ψ is a totally odd Hecke character for K of finite order. We can view ψ as a character of the absolute Galois group G_K. Let K_ψ be the corresponding cyclic extension of K and let $\Delta = \mathrm{Gal}(K_\psi/K)$. Then ψ becomes a faithful character of Δ and K_ψ is a CM field. Now let p be an odd prime. For simplicity, we will assume that the order of ψ divides $p-1$. We can then view ψ as a character with values in $\mathbf{Z}_p^\times$. Let D be the Galois module which is isomorphic to $\mathbf{Q}_p/\mathbf{Z}_p$ as a group and on which G_K acts by ψ. Let K_∞ denote the cyclotomic $\mathbf{Z}_p$-extension of K. Thus $\Gamma = \mathrm{Gal}(K_\infty/K)$ is isomorphic to $\mathbf{Z}_p$. Define $S(K_\infty, D)$ to be the subgroup of $H^1(K_\infty, D)$ consisting of everywhere unramified cocycle classes. As is usual in Iwasawa theory, we can view $S(K_\infty, D)$ as a discrete Λ-module, where $\Lambda = \mathbf{Z}_p[[\Gamma]]$ is the completed group algebra for Γ over $\mathbf{Z}_p$. Iwasawa's theorem asserts that the Pontryagin dual of $S(K_\infty, D)$ has no nonzero, finite Λ-submodules.

The Selmer group for the above Galois module D over K_∞, as defined in [Gr1], is precisely $S(K_\infty, D)$. Let $K_{\infty,\psi} = K_\psi K_\infty$, the cyclotomic $\mathbf{Z}_p$-extension of K_ψ. Under the restriction map $H^1(K_\infty, D) \to H^1(K_{\infty,\psi}, D)^\Delta$, one can identify $S(K_\infty, D)$ with $\mathrm{Hom}(X^{(\psi)}, D)$, where X is a certain Galois group on which Δ acts. To be precise, one takes $X = \mathrm{Gal}(L_\infty/K_{\infty,\psi})$, where L_∞ denotes the maximal, abelian pro-p extension of $K_{\infty,\psi}$ which is unramified at all primes of $K_{\infty,\psi}$. There is a canonical action of $G = \mathrm{Gal}(K_{\infty,\psi}/K)$ on X (defined by conjugation). Furthermore, we can identify Δ and Γ with $\mathrm{Gal}(K_{\infty,\psi}/K_\infty)$ and $\mathrm{Gal}(K_{\infty,\psi}/K_\psi)$,

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respectively, so that $G \cong \Delta \times \Gamma$. We define $X^{(\psi)}$ to be $e_\psi X$, where $e_\psi \in \mathbf{Z}_p[\Delta]$ is the idempotent for ψ. Iwasawa proved that $X^{(\psi)}$ has no nonzero, finite Λ-submodules. The theorem stated above is equivalent to that result.

To state the second classical result, suppose that K is any number field and that E is an elliptic curve defined over K with good, ordinary reduction at the primes of K lying above p. The p-primary subgroup $\mathrm{Sel}_E(K_\infty)_p$ of the Selmer group for E over K_∞ is again a discrete Λ-module. If $D = E[p^\infty]$, then $\mathrm{Sel}_E(K_\infty)_p$ can again be identified with the Selmer group for the Galois module D over K_∞ as defined in [Gr1]. Its Pontryagin dual $X_E(K_\infty)$ is a finitely-generated Λ-module. Mazur conjectured that $X_E(K_\infty)$ is a torsion Λ-module. If this is so, and if one makes the additional assumption that $E(K)$ has no element of order p, then one can show that $X_E(K_\infty)$ has no nonzero, finite Λ-submodule.

The above results take the following form: $\mathcal{S}$ is a certain discrete Λ-module. The Pontryagin dual $\mathcal{X} = \mathrm{Hom}(\mathcal{S}, \mathbf{Q}_p/\mathbf{Z}_p)$ is finitely generated as a Λ-module. The results assert that $\mathcal{X}$ has no nonzero finite Λ-submodule. An equivalent statement about $\mathcal{S}$ is the following: *There exists a nonzero element $\theta \in \Lambda$ such that $\pi\mathcal{S} = \mathcal{S}$ for all irreducible elements $\pi \in \Lambda$ not dividing θ.* We then say that $\mathcal{S}$ is an *"almost divisible"* Λ-module. Note that Λ is isomorphic to $\mathbf{Z}_p[[T]]$, a formal power series ring over $\mathbf{Z}_p$ in one variable, and hence is a unique factorization domain. Thus, one can equivalently say that $\lambda\mathcal{S} = \mathcal{S}$ for all $\lambda \in \Lambda$ which are relatively prime to θ. This definition makes sense in a much more general setting, as we now describe.

Suppose that Λ is isomorphic to a formal power series ring over $\mathbf{Z}_p$, or over $\mathbf{F}_p$, in a finite number of variables. Suppose that $\mathcal{S}$ is a discrete Λ-module and that its Pontryagin dual $\mathcal{X}$ is finitely generated. We then say that $\mathcal{S}$ is a cofinitely generated Λ-module. We say that $\mathcal{S}$ is an *almost divisible* Λ-module if any one of the five equivalent statements given below is satisfied. In the statements, the set of prime ideals of Λ of height 1 is denoted by $\mathrm{Spec}_{ht=1}(\Lambda)$. Note that since Λ is a UFD, all such prime ideals Π are principal. Also, if we say *almost all*, we mean *all but finitely many*. The notation $\mathcal{X}[\Pi]$ denotes the Λ-submodule of $\mathcal{X}$ consisting of elements annihilated by Π. This is also denoted by $\mathcal{X}[\pi]$, where π is a generator of Π. In the fifth statement, recall that a finitely-generated Λ-module $\mathcal{Z}$ is said to be pseudo-null if there exist two relatively prime elements of Λ which annihilate $\mathcal{Z}$.

- *We have $\Pi\mathcal{S} = \mathcal{S}$ for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.*
- *There exists a nonzero element θ in Λ such that $\pi\mathcal{S} = \mathcal{S}$ for all irreducible elements π of Λ not dividing θ.*
- *We have $\mathcal{X}[\Pi] = 0$ for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.*
- *The set $\mathrm{Ass}_\Lambda(\mathcal{Y})$ of associated prime ideals for the torsion Λ-submodule $\mathcal{Y}$ of $\mathcal{X}$ contains only prime ideals of height 1.*

- *The Λ-module $\mathcal{X}$ has no nonzero, pseudo-null submodules.*

We refer the reader to [Gr4] (and proposition 2.4, in particular) for further discussion, including an explanation of the equivalence of all of the above statements.

We will consider *Selmer groups* that arise in the following very general context. Suppose that K is a finite extension of $\mathbf{Q}$ and that Σ is a finite set of primes of K. Let K_Σ denote the maximal extension of K unramified outside of Σ. We assume that Σ contains all archimedean primes and all primes lying over some fixed rational prime p. The Selmer groups that we consider in this article are associated to a continuous representation

$$\rho : \mathrm{Gal}(K_\Sigma/K) \longrightarrow GL_n(R)$$

where R is a complete Noetherian local ring. Let $\mathfrak{M}$ denote the maximal ideal of R. We assume that the residue field $R/\mathfrak{M}$ is finite and has characteristic p. Hence R is compact in its $\mathfrak{M}$-adic topology. Let $\mathcal{T}$ be the underlying free R-module on which $\mathrm{Gal}(K_\Sigma/K)$ acts via ρ. We define $\mathcal{D} = \mathcal{T} \otimes_R \widehat{R}$, where $\widehat{R} = \mathrm{Hom}(R, \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontryagin dual of R with a trivial action of $\mathrm{Gal}(K_\Sigma/K)$. That Galois group acts on $\mathcal{D}$ through its action on the first factor $\mathcal{T}$. Thus, $\mathcal{D}$ is a discrete abelian group which is isomorphic to $\widehat{R}^n$ as an R-module and which has a continuous R-linear action of $\mathrm{Gal}(K_\Sigma/K)$.

The Galois cohomology group $H^1(K_\Sigma/K, \mathcal{D})$ can be considered as a discrete R-module too. It is a cofinitely generated R-module in the sense that its Pontryagin dual is finitely generated as an R-module. (See Prop. 3.2 in [Gr4].) Suppose that one specifies an R-submodule $L(K_v, \mathcal{D})$ of $H^1(K_v, \mathcal{D})$ for each $v \in \Sigma$. We will denote such a specification by $\mathcal{L}$ for brevity. Let

$$P(K, \mathcal{D}) \;=\; \prod_{v \in \Sigma} H^1(K_v, \mathcal{D}) \qquad \textit{and} \qquad L(K, \mathcal{D}) \;=\; \prod_{v \in \Sigma} L(K_v, \mathcal{D}) \quad .$$

Thus, $L(K, \mathcal{D})$ is an R-submodule of $P(K, \mathcal{D})$. Let $Q_{\mathcal{L}}(K, \mathcal{D}) = P(K, \mathcal{D})/L(K, \mathcal{D})$. Thus,

$$Q_{\mathcal{L}}(K, \mathcal{D}) \;=\; \prod_{v \in \Sigma} Q_{\mathcal{L}}(K_v, \mathcal{D}), \qquad \textit{where} \quad Q_{\mathcal{L}}(K_v, \mathcal{D}) \;=\; H^1(K_v, \mathcal{D})/L(K_v, \mathcal{D}) \quad .$$

The natural global-to-local restriction maps for $H^1(\,\cdot\,, \mathcal{D})$ induce a map

$$(1) \qquad \phi_{\mathcal{L}} : \; H^1(K_\Sigma/K, \mathcal{D}) \longrightarrow Q_{\mathcal{L}}(K, \mathcal{D}) \quad .$$

The kernel of $\phi_{\mathcal{L}}$ will be denoted by $S_{\mathcal{L}}(K, \mathcal{D})$. It is the *"Selmer group"* for $\mathcal{D}$ over K corresponding to the specification $\mathcal{L}$.

It is clear that $S_{\mathcal{L}}(K,\mathcal{D})$ is an R-submodule of $H^1(K_\Sigma/K,\mathcal{D})$ and so is also a discrete, cofinitely generated R-module. For a fixed set Σ, the smallest possible Selmer group occurs when we take $L(K_v,\mathcal{D})=0$ for all $v\in\Sigma$. The Selmer group corresponding to that choice will be denoted by $Ш^1(K,\Sigma,\mathcal{D})$. That is,

$$Ш^1(K,\Sigma,\mathcal{D}) \;=\; \ker\Big(H^1(K_\Sigma/K,\mathcal{D}) \longrightarrow \prod_{v\in\Sigma} H^1(K_v,\mathcal{D})\Big)$$

Obviously, we have $Ш^1(K,\Sigma,\mathcal{D}) \subseteq S_{\mathcal{L}}(K,\mathcal{D})$ for any choice of the specification $\mathcal{L}$.

In addition to the above assumptions about R, suppose that R is a domain. Let $d=m+1$ denote the Krull dimension of R, where $m\geq 0$. (We will assume that R is not a field. Our results are all trivial in that case.) A theorem of Cohen [Coh] implies that R is a finite, integral extension of a subring Λ which is isomorphic to one of the formal power series rings $\mathbf{Z}_p[[T_1,...,T_m]]$ or $\mathbf{F}_p[[T_1,...,T_{m+1}]]$, depending on whether R has characteristic 0 or p. Although such a subring is far from unique, it will be convenient to just fix a choice. A cofinitely generated R-module $\mathcal{S}$ will also be a cofinitely generated Λ-module. All the results that we will prove in this paper could be viewed as statements about the structure of the Selmer groups as R-modules. But they are equivalent to the corresponding statements about their structure as Λ-modules and that is how we will formulate and prove them. Those equivalences are discussed in some detail in [Gr4], section 2. In particular, if $\mathcal{S}$ is a discrete, cofinitely generated R-module, then we say that $\mathcal{S}$ is divisible (resp., almost divisible) as an R-module if $P\mathcal{S}=\mathcal{S}$ for all (resp., almost all) $P\in \mathrm{Spec}_{ht=1}(R)$. One result is that $\mathcal{S}$ is almost divisible as an R-module if and only if $\mathcal{S}$ is almost divisible as a Λ-module. (See statement **1** on page 350 of [Gr4].) A similar equivalence is true for divisibility, but quite easy to prove.

One basic assumption that we will make about R is that it contain a subring Λ of the form described in the previous paragraph, that R is finitely-generated as a Λ-module, and that R is also reflexive as a Λ-module. If these assumptions are satisfied, we say that R is a *"reflexive ring"*. In the case where R is also assumed to be a domain, one can equivalently require that R is the intersection of all its localizations at prime ideals of height 1. See part D, section 2 in [Gr4] for the explanation of the equivalence. In the literature, one sometimes finds the term *"weakly Krull domain"* for such a domain. The class of reflexive domains is rather large. For example, if R is integrally closed or Cohen-Macaulay, then it turns out that R is reflexive. There are important examples (from Hida theory), where R is not necessarily a domain, but is still a free (and hence reflexive) module over a suitable subring Λ.

The main results of this paper assert that if we make certain hypotheses about $\mathcal{D}$ and $\mathcal{L}$, then $S_{\mathcal{L}}(K,\mathcal{D})$ will be almost divisible. Some of the hypotheses are those needed for theorem 1 in [Gr4] which gives sufficient conditions for $H^1(K_\Sigma/K,\mathcal{D})$ itself to be almost divisible.

That theorem will be stated later (as proposition 2.6.1.) and is our starting point. The basic approach for deducing the almost divisibility of a Λ-submodule of $H^1(K_\Sigma/K, \mathcal{D})$, defined by imposing local conditions corresponding to a specification $\mathcal{L}$, will be described in section 3. Some of the needed hypotheses will be discussed in section 2. We also state there some results from [Gr5] concerning the surjectivity of the map $\phi_{\mathcal{L}}$. We will apply those results not just to $\mathcal{D}$, but also to the corresponding map for $\mathcal{D}[\Pi]$, where $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Our main results concerning the almost divisibility of $S_{\mathcal{L}}(K, \mathcal{D})$ will be proved in section 4.1. We show in section 4.2 how to prove the classical theorems mentioned above from the point of view of this paper.

This paper is part of a series of papers concerning foundational questions in Iwasawa theory. The results discussed above depend on the results proved in [Gr4] and [Gr5], the first papers in this series. A subsequent paper will use the results we prove here to study the behavior of Selmer groups under specialization. In particular, one would like to understand how the *"characteristic ideal"* or *"characteristic divisor"* for a Selmer group associated to the representation $\rho_P : \mathrm{Gal}(K_\Sigma/K) \longrightarrow GL_n(R/P)$, the reduction of ρ modulo a prime ideal P of R, is related to the characteristic ideal or divisor associated to a Selmer group for ρ itself. Such a question has arisen many times in the past. Consequently, for the purpose of studying exactly that question, one can find numerous special cases of the results of this paper in the literature on Iwasawa theory.

2 Various Hypotheses.

The R-module $\mathcal{T}$ is a free R-module and so we say that $\mathcal{D}$ is a cofree R-module. We also define $\mathcal{T}^* = \mathrm{Hom}(\mathcal{D}, \mu_{p^\infty})$. We can consider $\mathcal{T}^*$ as a module over the ring R^{op}, which is just R since that ring is commutative. It is clear that $\mathcal{T}^*$ is also a free R-module and that the discrete R-module $\mathcal{D}^* = \mathcal{T}^* \otimes_R \widehat{R}$ is cofree. It will be simpler and more useful to formulate the hypotheses in terms of their structure as Λ-modules rather than R-modules.

2.1. Hypotheses involving reflexivity. Recall that Λ is isomorphic to a formal power series ring in a finite number of variables over either $\mathbf{Z}_p$ or $\mathbf{F}_p$. Reflexive Λ-modules play an important role here. A detailed discussion of the definition can be found in section 2, part C, of [Gr4]. We almost always will assume the following hypothesis.

RFX($\mathcal{D}$): *The Λ-module $\mathcal{T}$ is reflexive.*

Equivalently, since $\mathcal{T}$ is free as an R-module, RFX($\mathcal{D}$) means that the ring R is reflexive as a Λ-module. That is, R is a reflexive ring in the sense defined in the introduction. We are still always assuming that R is a complete Noetherian local ring with finite residue field of characteristic p.

We will say that $\mathcal{D}$ is a coreflexive Λ-module if $\mathrm{RFX}(\mathcal{D})$ holds. This terminology is appropriate because $\mathcal{D}$ is isomorphic to the Λ-module $\widehat{R}^n$ (ignoring the Galois action) and its Pontryagin dual is the reflexive Λ-module R^n. One important role of $\mathrm{RFX}(\mathcal{D})$ is to guarantee that $\mathcal{D}[\pi]$ is a divisible (Λ/Π)-module for all prime ideals $\Pi = (\pi)$ in Λ of height 1. That property is equivalent to requiring that $\mathcal{D}$ be coreflexive as a Λ-module. See corollary 2.6.1 in [Gr4] for the proof.

The next two hypotheses involve $\mathcal{T}^*$ and are of a local nature. They could be formulated just in terms of $\mathcal{D}$, but the statements would become more complicated. Note that if $\mathrm{RFX}(\mathcal{D})$ holds, then $\mathcal{T}^*$ is also a reflexive Λ-module. We suppose that v is a prime of K and that K_v is the completion of K at v. We usually consider just the primes $v \in \Sigma$.

$\mathrm{LOC}_v^{(1)}(\mathcal{D})$: $\quad (\mathcal{T}^*)^{G_{K_v}} = 0$.

$\mathrm{LOC}_v^{(2)}(\mathcal{D})$: *The* Λ*-module* $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ *is reflexive.*

Assumptions $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ play a crucial role in proving theorem 1 in [Gr4]. Just as in that result, we will usually assume $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ for at least one non-archimedean prime $v \in \Sigma$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ for all $v \in \Sigma$. One can find a general discussion of when those hypotheses are satisfied in part F, section 5 of [Gr4]. One obvious remark is that since $\mathcal{T}^*$ is a torsion-free Λ-module, $\mathrm{LOC}_v^{(1)}$ is satisfied if and only if $\mathrm{rank}_\Lambda\big((\mathcal{T}^*)^{G_{K_v}}\big) = 0$. It is also obvious that $\mathcal{T}^*/(\mathcal{T}^*)^{G_{K_v}}$ is at least torsion-free as a Λ-module. Furthermore, note that if $\mathrm{RFX}(\mathcal{D})$ is true, then $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ follows from $\mathrm{LOC}_v^{(1)}(\mathcal{D})$. Notice also that if $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ are both true for some prime v, then $\mathrm{RFX}(\mathcal{D})$ is also true. Nevertheless, our propositions will often include $\mathrm{RFX}(\mathcal{D})$ as a hypothesis even though it may actually be implied by other hypotheses.

2.2. Locally trivial cocycle classes. The following much more subtle hypothesis is also needed in the proof of theorem 1 in [Gr4], where it is referred to as Hypothesis L. As we explain there, it can be viewed as a generalization of Leopoldt's Conjecture for number fields. That special case occurs when $\Lambda = \mathbf{Z}_p$, $\mathcal{D} = \mathbf{Q}_p/\mathbf{Z}_p$, and G_K acts trivially on $\mathcal{D}$. For the formulation, we define

$$Ш^2(K, \Sigma, \mathcal{D}) \;=\; \ker\Big(H^2(K_\Sigma/K, \mathcal{D}) \longrightarrow \prod_{v \in \Sigma} H^2(K_v, \mathcal{D})\Big),$$

which is a discrete, cofinitely-generated Λ-module.

LEO($\mathcal{D}$): *The Λ-module $Ш^2(K, \Sigma, \mathcal{D})$ is cotorsion.*

A long discussion about the validity of the above hypothesis can be found in the last few pages of section 6, part D, in [Gr4]. There are situations where it fails to be satisfied. Also, section 4 of that paper derives a natural lower bound on the Λ-corank of $H^1(K_\Sigma/K, \mathcal{D})$ from the duality theorems of Poitou and Tate. Hypothesis LEO($\mathcal{D}$) is equivalent to the statement that $\mathrm{corank}_\Lambda\big(H^1(K_\Sigma/K, \mathcal{D})\big)$ is equal to that lower bound. That equivalence is the content of propositions 4.3 and 4.4 in [Gr4]. Furthermore, one part of theorem 1 in that paper asserts that if RFX($\mathcal{D}$) is satisfied, and if we assume $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ for at least one non-archimedean prime $v \in \Sigma$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ for all $v \in \Sigma$, then LEO($\mathcal{D}$) means that $Ш^2(K, \Sigma, \mathcal{D})$ actually vanishes.

2.3. Hypotheses involving $\mathcal{L}$. None of the hypotheses stated above involves the specification $\mathcal{L}$. We now mention two hypotheses which do involve $\mathcal{L}$, one of which implies the other. They are statements about the cokernel of the map $\phi_{\mathcal{L}}$ defined in the introduction. The first plays an important role in studying Selmer groups. The second appears weaker, but often is sufficient to imply the first.

SUR($\mathcal{D}, \mathcal{L}$): *The map $\phi_{\mathcal{L}}$ defining $S_{\mathcal{L}}(K, \mathcal{D})$ is surjective.*

An obvious necessary condition for this to be satisfied is the following equality for the coranks:

$$\mathrm{CRK}(\mathcal{D}, \mathcal{L}): \quad \mathrm{corank}_\Lambda\big(H^1(K_\Sigma/K, \mathcal{D})\big) = \mathrm{corank}_\Lambda\big(S_{\mathcal{L}}(K, \mathcal{D})\big) + \mathrm{corank}_\Lambda\big(Q_{\mathcal{L}}(K, \mathcal{D})\big) .$$

This just means that $\mathrm{coker}\big(\phi_{\mathcal{L}}\big)$ is a cotorsion Λ-module. Proposition 3.2.1 in [Gr5] shows that CRK($\mathcal{D}, \mathcal{L}$), together with various additional assumptions, actually implies SUR($\mathcal{D}, \mathcal{L}$). One has the following obvious inequality:

$$\mathrm{corank}_\Lambda\big(S_{\mathcal{L}}(K, \mathcal{D})\big) \geq \mathrm{corank}_\Lambda\big(H^1(K_\Sigma/K, \mathcal{D})\big) - \mathrm{corank}_\Lambda\big(Q_{\mathcal{L}}(K, \mathcal{D})\big) \tag{2}$$

Thus, CRK($\mathcal{D}, \mathcal{L}$) is equivalent to having equality here. Of course, CRK($\mathcal{D}, \mathcal{L}$), and hence SUR($\mathcal{D}, \mathcal{L}$), can fail simply because the quantity on the right side is negative. Verifying CRK($\mathcal{D}, \mathcal{L}$) is quite a difficult problem in many interesting cases.

It is worth recalling what the formulas for global and local Euler-Poincaré characteristics tell us about the coranks on the right side of (2). One can find proofs in section 4 of [Gr4]. For any prime v of K, we use the notation $\mathcal{D}(K_v)$ as an abbreviation for $H^0(K_v, \mathcal{D})$, a Λ-submodule of $\mathcal{D}$. Similarly, $\mathcal{D}(K)$ will denote $H^0(K, \mathcal{D})$. Let $r_1(K)$ and $r_2(K)$ denote the number of real primes and complex primes of K, respectively. We give formulas for the Λ-coranks of the global and local H^1's. For the global H^1, we have

$$\mathrm{corank}_\Lambda\big(H^1(K_\Sigma/K,\mathcal{D})\big) \;=\; \mathrm{corank}_\Lambda\big(\mathcal{D}(K)\big) \;+\; \mathrm{corank}_\Lambda\big(H^2(K_\Sigma/K,\mathcal{D})\big) \;+\; \delta_\Lambda(K,\mathcal{D}) \;\;,$$

where $\delta_\Lambda(K,\mathcal{D}) = \big(r_1(K)+r_2(K)\big)\mathrm{corank}_\Lambda\big(\mathcal{D}\big) - \sum_{v\ real}\mathrm{corank}_\Lambda\big(\mathcal{D}(K_v)\big)$.

Now assume that v is a non-archimedean prime. Recall that $\mathcal{D}^*$ denotes $\mathcal{T}^*\otimes_R \widehat{R}$. If v does not lie over p, then the local Euler-Poincaré characteristic is 0 and we have

$$\mathrm{corank}_\Lambda\big(H^1(K_v,\mathcal{D})\big) \;=\; \mathrm{corank}_\Lambda\big(\mathcal{D}(K_v)\big) \;+\; \mathrm{corank}_\Lambda\big(\mathcal{D}^*(K_v)\big) \quad .$$

To justify replacing the Λ-corank of $H^2(K_v,\mathcal{D})$ by that of $\mathcal{D}^*(K_v)$ in the above formula as well as the formula below, one uses the fact that the Pontryagin dual of $H^2(K_v,\mathcal{D})$ is isomorphic to $H^0(K_v,\mathcal{T}^*)$. Proposition 3.10 in [Gr4] implies that the Λ-rank of $H^0(K_v,\mathcal{T}^*)$ is equal to the Λ-corank of $H^0(K_v,\mathcal{D}^*)$. If v lies over p, then we have

$$\mathrm{corank}_\Lambda\big(H^1(K_v,\mathcal{D})\big) \;=\; \mathrm{corank}_\Lambda\big(\mathcal{D}(K_v)\big) \;+\; \mathrm{corank}_\Lambda\big(\mathcal{D}^*(K_v)\big) \;+\; [K_v:\mathbf{Q}_p]\mathrm{corank}_\Lambda\big(\mathcal{D}\big) \quad .$$

If v is archimedean, then $H^1(K_v,\mathcal{D})$ vanishes unless $p=2$ and v is real. Even for $p=2$, its Λ-corank is 0 unless Λ is a power series ring over $\mathbf{F}_2$. In that case, one has the following formula when v is real:

$$\mathrm{corank}_\Lambda\big(H^1(K_v,\mathcal{D})\big) \;=\; 2\mathrm{corank}_\Lambda\big(\mathcal{D}(K_v)\big) \;-\; n \quad .$$

Here $n=\mathrm{corank}_\Lambda\big(\mathcal{D}\big)$. See page 380 of [Gr4] for the simple justification.

Finally, we have the obvious formula

$$\mathrm{corank}_\Lambda\big(Q_{\mathcal{L}}(K_v,\mathcal{D})\big) \;=\; \mathrm{corank}_\Lambda\big(H^1(K_v,\mathcal{D})\big) \;-\; \mathrm{corank}_\Lambda\big(L(K_v,\mathcal{D})\big)$$

and so the above formulas for the Λ-coranks of $H^1(K_v,\mathcal{D})$ for $v\in\Sigma$, and the specification $\mathcal{L}$, determine the Λ-corank of $Q_{\mathcal{L}}(K,D)$.

2.4. Behavior under specialization. In some proofs, Selmer groups for $\mathcal{D}[\Pi]$, as well as for $\mathcal{D}$, will occur. Here Π is a prime ideal of Λ and $\mathcal{D}[\Pi]$ is a discrete, cofinitely-generated module over the ring Λ/Π. Various other modules over Λ/Π will arise. Now Λ/Π is a complete, Noetherian, local ring, and therefore (just as for R in the introduction), it is a finite, integral extension of a subring Λ' which is isomorphic to a formal power series ring over $\mathbf{Z}_p$ or $\mathbf{F}_p$. We fix such a choice for each Π and denote Λ' by Λ_Π. If Λ has Krull dimension d, then Λ_Π has Krull dimension $d-1$. Of course, some results could be easily stated or proved just in terms of Λ/Π itself.

Many of the above hypotheses are not preserved when the Λ-module $\mathcal{D}$ is replaced by the Λ_Π-module $\mathcal{D}[\Pi]$. For example, even if $\mathrm{RFX}(\mathcal{D})$ is satisfied, $\mathcal{D}[\Pi]$ may fail to be reflexive as a Λ_Π-module and so $\mathrm{RFX}(\mathcal{D}[\Pi])$ may fail to be satisfied. In general, all one can say is that $\mathrm{RFX}(\mathcal{D})$ implies that $\mathcal{D}[\Pi]$ is a divisible Λ_Π-module for all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. The situation is better for $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ and $\mathrm{LEO}(\mathcal{D})$. We have the following equivalences.

- *Assume that* $\mathrm{RFX}(\mathcal{D})$ *is satisfied. Then* $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ *is true if and only if* $\mathrm{LOC}_v^{(1)}(\mathcal{D}[\Pi])$ *is true for almost all* $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

- $\mathrm{LEO}(\mathcal{D})$ *is true if and only if* $\mathrm{LEO}(\mathcal{D}[\Pi])$ *is true for almost all* $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

These assertions follow easily from results in [Gr4]. For the first statement, one should see remarks 3.5.1 or 3.10.2 there. Note that $\mathrm{LOC}_v^{(1)}(\mathcal{D}[\Pi])$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D}[\Pi])$ are statements about the (Λ/Π)-module $\mathrm{Hom}(\mathcal{D}[\Pi], \mu_{p^\infty})$, which is isomorphic to $\mathcal{T}^*/\Pi\mathcal{T}^*$. The second of the above equivalences follows from lemma 4.4.1 and remark 2.1.3 in [Gr4]. We will prove a similar equivalence for $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ in section 3.4.

2.5. A result about almost divisibility. In addition to $\mathrm{SUR}(\mathcal{D}, \mathcal{L})$ and $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$, there will be various other hypotheses concerning the specification $\mathcal{L}$. If $L(K_v, \mathcal{D})$ is Λ-divisible (resp., almost Λ-divisible) for all $v \in \Sigma$, then we will say that $\mathcal{L}$ is Λ-divisible (resp, almost Λ-divisible). Consider another specification $\mathcal{L}'$ for $\mathcal{D}$ and let $L'(K_v, \mathcal{D})$ denote the corresponding subgroup of $H^1(K_v, \mathcal{D})$ for each $v \in \Sigma$. We write $\mathcal{L}' \subseteq \mathcal{L}$ if $L'(K_v, \mathcal{D}) \subseteq L(K_v, \mathcal{D})$ for all $v \in \Sigma$. In particular, if $L'(K_v, \mathcal{D}) = L(K_v, \mathcal{D})_{\Lambda\text{-}div}$ for each $v \in \Sigma$, then we will refer to the specification $\mathcal{L}'$ as the maximal Λ-divisible subspecification of $\mathcal{L}$, which we denote simply by $\mathcal{L}_{div}$. One assumption that we will usually make is that $\mathcal{L}$ is almost divisible. Its importance is clear from the following proposition.

Proposition 2.5.1. *Assume that* $\mathcal{L}'$ *and* $\mathcal{L}$ *are specifications for* $\mathcal{D}$ *and that* $\mathcal{L}' \subseteq \mathcal{L}$. *Assume also that* $\mathrm{SUR}(\mathcal{D}, \mathcal{L}')$ *is true. Then* $\mathrm{SUR}(\mathcal{D}, \mathcal{L})$ *is also true,* $S_{\mathcal{L}'}(K, \mathcal{D}) \subseteq S_{\mathcal{L}}(K, \mathcal{D})$, *and*

$$S_{\mathcal{L}}(K, \mathcal{D})/S_{\mathcal{L}'}(K, \mathcal{D}) \;\cong\; \prod_{v \in \Sigma} L(K_v, \mathcal{D})/L'(K_v, \mathcal{D})$$

as Λ*-modules. In particular, if* $\mathrm{SUR}(\mathcal{D}, \mathcal{L}_{div})$ *is true and* $S_{\mathcal{L}_{div}}(K, \mathcal{D})$ *is almost* Λ*-divisible, then* $S_{\mathcal{L}}(K, \mathcal{D})$ *is almost* Λ*-divisible if and only if* $\mathcal{L}$ *is almost* Λ*-divisible. If* $\mathrm{SUR}(\mathcal{D}, \mathcal{L}_{div})$ *is true and* $S_{\mathcal{L}}(K, \mathcal{D})$ *is almost* Λ*-divisible, then* $\mathcal{L}$ *must be almost* Λ*-divisible.*

Thus, under certain assumptions, the structure of $S_{\mathcal{L}}(K, \mathcal{D})$ can be related to that of $S_{\mathcal{L}_{div}}(K, \mathcal{D})$ and the quotient Λ-modules $L(K_v, \mathcal{D})/L(K_v, \mathcal{D})_{div}$ for $v \in \Sigma$. Since all of those quotients are cofinitely-generated, cotorsion Λ-module for all $v \in \Sigma$, it follows that $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ is true if and only if $\mathrm{CRK}(\mathcal{D}, \mathcal{L}_{div})$ is true.

Proof. Most of the statements are clear from the definitions. For the isomorphism, consider the following maps:

$$H^1(K_\Sigma/K, \mathcal{D}) \xrightarrow{\phi_{\mathcal{L}'}} Q_{\mathcal{L}'}(K, \mathcal{D}) \xrightarrow{\psi} Q_{\mathcal{L}}(K, \mathcal{D})$$

where ψ is the natural map, the canonical homomorphism whose kernel is the direct product in the proposition. The map ψ is surjective and the composition is $\phi_{\mathcal{L}}$. If $\phi_{\mathcal{L}'}$ is surjective, then it follows that $\phi_{\mathcal{L}}$ is also surjective and that $S_{\mathcal{L}}(K, \mathcal{D})\big/S_{\mathcal{L}'}(K, \mathcal{D})$ is isomorphic to $\ker(\psi)$. The stated isomorphism follows immediately. For the final statements, one takes $\mathcal{L}' = \mathcal{L}_{div}$. Note that if $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible, and if one assumes $\mathrm{SUR}(\mathcal{D}, \mathcal{L}_{div})$, then there is a surjective homomorphism from $S_{\mathcal{L}}(K, \mathcal{D})$ to $L(K_v, \mathcal{D})\big/L(K_v, \mathcal{D})_{\Lambda\text{-}div}$, which must therefore be almost divisible too. This implies that $L(K_v, \mathcal{D})$ is then almost divisible for all $v \in \Sigma$. Thus, $\mathcal{L}$ is almost divisible. Moreover, if a discrete, cofinitely-generated Λ-module $\mathcal{S}$ contains an almost divisible Λ-submodule $\mathcal{S}'$, then it is clear that $\mathcal{S}$ is almost divisible if and only if $\mathcal{S}/\mathcal{S}'$ is almost divisible. ■

2.6. The main results in [Gr4] and [Gr5]. The following result is proved in [Gr4] . It is part of the theorem 1 which we alluded to before. It plays a crucial role in this paper because we will study when $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible as a Λ-module under the assumption that $H^1(K_\Sigma/K, \mathcal{D})$ is almost divisible, as outlined in the next section.

Proposition 2.6.1. *Suppose that* $\mathrm{RFX}(\mathcal{D})$ *and* $\mathrm{LEO}(\mathcal{D})$ *are satisfied, that* $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ *is satisfied for all* v *in* Σ*, and that there exists a non-archimedean prime* $\eta \in \Sigma$ *such that* $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ *is satisfied. Then* $H^1(K_\Sigma/K, \mathcal{D})$ *is an almost divisible* Λ*-module.*

Another part of theorem 1 is the following.

Proposition 2.6.2. *Suppose that* $\mathrm{RFX}(\mathcal{D})$ *is satisfied, that* $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ *is satisfied for all* v *in* Σ*, and that there exists a non-archimedean prime* $\eta \in \Sigma$ *such that* $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ *is satisfied. Then* $\text{Ш}^2(K, \Sigma, \mathcal{D})$ *is a coreflexive* Λ*-module.*

The conclusion in this result has the interesting consequence that the Pontryagin dual of $\text{Ш}^2(K, \Sigma, \mathcal{D})$ is torsion-free as a Λ-module. It follows that $\text{Ш}^2(K, \Sigma, \mathcal{D})$ is Λ-divisible. Hence either $\text{Ш}^2(K, \Sigma, \mathcal{D})$ has positive Λ-corank or $\text{Ш}^2(K, \Sigma, \mathcal{D}) = 0$ under the assumptions in proposition 2.6.2.

We now state the main result that we need from [Gr5]. It is proposition 3.2.1 there.

Proposition 2.6.3. *Suppose that $\mathcal{D}$ is divisible as a Λ-module. Assume that* $\mathrm{LEO}(\mathcal{D})$*,* $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$*, and also at least one of the following additional assumptions is satisfied.*

(a) $\mathcal{D}[\mathfrak{m}]$ has no subquotient isomorphic to μ_p for the action of G_K,

(b) $\mathcal{D}$ is a cofree Λ-module and $\mathcal{D}[\mathfrak{m}]$ has no quotient isomorphic to μ_p for the action of G_K,

(c) There is a prime $\eta \in \Sigma$ satisfying the following properties: (i) $H^0(K_\eta, \mathcal{T}^*) = 0$, and (ii) $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ is divisible as a Λ-module.

Then $\phi_{\mathcal{L}}$ is surjective.

As mentioned in the introduction, we will apply the above result not just to $\mathcal{D}$, but also to $\mathcal{D}[\Pi]$ for prime ideals Π of Λ of height 1. Fortunately, if $\mathcal{D}$ is itself coreflexive as a Λ-module, then $\mathcal{D}[\Pi]$ is divisible as a (Λ/Π)-module, and hence satisfies the first hypothesis in the above proposition.

3 An Outline.

3.1. An exact sequence. Assume that $\mathrm{SUR}(\mathcal{D}, \mathcal{L})$ is satisfied. We will denote $\phi_{\mathcal{L}}$ just by ϕ, although we will continue to indicate the $\mathcal{L}$ for other objects. We have an exact sequence

$$0 \longrightarrow S_{\mathcal{L}}(K, \mathcal{D}) \longrightarrow H^1(K_\Sigma/K, \mathcal{D}) \xrightarrow{\phi} Q_{\mathcal{L}}(K, \mathcal{D}) \longrightarrow 0 \tag{3}$$

of discrete Λ-modules. Suppose that $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$ and that π is a generator of Π. Applying the snake lemma to the exact sequence (3) and to the endomorphisms of each of the above modules induced by multiplication by π, we obtain the following important exact sequence. We refer to it as the *snake lemma sequence for* Π.

$$H^1(K_\Sigma/K, \mathcal{D})[\Pi] \xrightarrow{\alpha_\Pi} Q_{\mathcal{L}}(K, \mathcal{D})[\Pi] \longrightarrow$$

$$\longrightarrow S_{\mathcal{L}}(K, \mathcal{D})/\Pi S_{\mathcal{L}}(K, \mathcal{D}) \longrightarrow H^1(K_\Sigma/K, \mathcal{D})/\Pi H^1(K_\Sigma/K, \mathcal{D})$$

Now assume additionally that $H^1(K_\Sigma/K, \mathcal{D})$ is an almost divisible Λ-module. The last term in the above exact sequence is then trivial for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Therefore, under these assumptions, the assertion that $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible is equivalent to the assertion that α_Π is surjective for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. We study the surjectivity of α_Π by considering the (Λ/Π)-module $\mathcal{D}[\pi]$.

3.2. The cokernel of α_Π. If $\mathcal{D}$ arises from a representation ρ as described in the introduction, and if R is a domain, then $\mathcal{D}$ will be Λ-divisible. However, here we will just assume that $\mathcal{D}$ is divisible by π and cofinitely generated as a Λ-module. We then have an exact sequence

$$0 \longrightarrow \mathcal{D}[\Pi] \longrightarrow \mathcal{D} \longrightarrow \mathcal{D} \longrightarrow 0$$

induced by multiplication by π. As a consequence, the following global and local *"specialization"* maps are surjective:

$$h_\Pi : H^1(K_\Sigma/K, \mathcal{D}[\Pi]) \longrightarrow H^1(K_\Sigma/K, \mathcal{D})[\Pi], \qquad h_{\Pi,v} : \ H^1(K_v, \mathcal{D}[\Pi]) \longrightarrow H^1(K_v, \mathcal{D})[\Pi]$$

We can compare the exact sequence (3) with an analogous sequence for $\mathcal{D}[\Pi]$, viewed as a (Λ/Π)-module. For this purpose, we define a specification $\mathcal{L}_\Pi$ for $\mathcal{D}[\Pi]$ as follows: For each $v \in \Sigma$, let us take

$$L(K_v, \mathcal{D}[\Pi]) \ = \ h_{\Pi,v}^{-1}\big(\ L(K_v, \mathcal{D})[\Pi] \ \big)$$

which is a (Λ/Π)-submodule of $H^1(K_v, \mathcal{D}[\Pi])$. If we think of $\mathcal{L}$ as fixed, we will refer to the specification $\mathcal{L}_\Pi$ just defined as the *"$\mathcal{L}$-maximal specification for $\mathcal{D}[\Pi]$"*. Using the analogous notation to that for $\mathcal{D}$, we define

$$P(K, \mathcal{D}[\Pi]) = \prod_{v \in \Sigma} H^1(K_v, \mathcal{D}[\Pi]), \qquad Q_{\mathcal{L}_\Pi}(K, \mathcal{D}[\Pi]) = P(K, \mathcal{D}[\Pi])\big/L(K, \mathcal{D}[\Pi])$$

where $L(K, \mathcal{D}[\Pi]) = \prod_{v\in\Sigma} L(K_v, \mathcal{D}[\Pi])$. We can then define the corresponding global-to-local map

$$\phi_{\mathcal{L}_\Pi} : \ H^1(K_\Sigma/K, \mathcal{D}[\Pi]) \ \longrightarrow \ Q_{\mathcal{L}_\Pi}(K, \mathcal{D}[\Pi])$$

We will usually denote the map $\phi_{\mathcal{L}_\Pi}$ simply by ϕ_Π. The product of the $h_{\Pi,v}$'s for $v \in \Sigma$ defines a map $b_\Pi : \ P(K, \mathcal{D}[\Pi]) \to P(K, \mathcal{D})[\Pi]$. The image of $L(K, \mathcal{D}[\Pi])$ under b_Π is contained in $L(K, \mathcal{D})$ and so we get a well-defined map

$$q_\Pi : \ Q_{\mathcal{L}_\Pi}(K, \mathcal{D}[\Pi]) \ \longrightarrow \ Q_{\mathcal{L}}(K, \mathcal{D})[\Pi] \ .$$

Lemma 3.2.1. *Assume that $\mathcal{D}$ is divisible by π and that $L(K_v, \mathcal{D})$ is divisible by π for all $v \in \Sigma$. Then q_Π is an isomorphism.*

Proof. The definition of $\mathcal{L}_\Pi$ implies that q_Π is injective without any assumptions. Furthermore, a snake lemma argument shows that if $L(K,\mathcal{D})$ is divisible by π, then the map $c_\Pi: P(K,\mathcal{D})[\Pi] \to Q_\mathcal{L}(K,\mathcal{D})[\Pi]$ will be surjective. Since the map b_Π is also surjective, it would then follow that $c_\Pi \circ b_\Pi$ is also surjective. This would imply that q_Π is surjective. ■

Consequently, if we assume that $\mathcal{D}$ is almost Λ-divisible and that $\mathcal{L}$ is almost Λ-divisible, we see that q_Π is then an isomorphism for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

The map α_Π is induced by the map ϕ and is defined without making any assumptions. We have the following commutative diagram whose rows are exact:

$$\begin{array}{ccccccc}
0 \longrightarrow & S_{\mathcal{L}_\Pi}(K,\mathcal{D}[\Pi]) & \longrightarrow & H^1(K_\Sigma/K,\mathcal{D}[\Pi]) & \xrightarrow{\phi_\Pi} & Q_{\mathcal{L}_\Pi}(K,\mathcal{D}[\Pi]) \\
& \downarrow{\scriptstyle s_\Pi} & & \downarrow{\scriptstyle h_\Pi} & & \downarrow{\scriptstyle q_\Pi} \\
0 \longrightarrow & S_\mathcal{L}(K,\mathcal{D})[\Pi] & \longrightarrow & H^1(K_\Sigma/K,\mathcal{D})[\Pi] & \xrightarrow{\alpha_\Pi} & Q_\mathcal{L}(K,\mathcal{D})[\Pi]
\end{array}$$

The second and third vertical maps have been defined and make that part of the diagram commutative, and so the map s_Π is induced from h_Π. Although it is not needed now, we remark in passing that the injectivity of the map q_Π and the surjectivity of the map h_Π imply that s_Π is also surjective. But the important consequence for us is that q_Π maps $\mathrm{im}(\phi_\Pi)$ isomorphically to $\mathrm{im}(\alpha_\Pi)$ and therefore induces an isomorphism

$$\mathrm{coker}(\alpha_\Pi) \;\cong\; \mathrm{coker}(\phi_\Pi) \tag{4}$$

under the assumptions in lemma 3.2.1. In particular, the surjectivity of α_Π and ϕ_Π would then be equivalent,

To summarize, if we assume that $\mathcal{D}$, $H^1(K_\Sigma/K,\mathcal{D})$, and the specification $\mathcal{L}$ are almost Λ-divisible, and that $\mathrm{SUR}(\mathcal{D},\mathcal{L})$ holds, then $S_\mathcal{L}(K,\mathcal{D})$ is almost Λ-divisible if and only if ϕ_Π is surjective for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

Remark 3.2.2. *Divisibility by* $\mathfrak{m}$. One can ask if $\mathfrak{m}S_\mathcal{L}(K,\mathcal{D}) = S_\mathcal{L}(K,\mathcal{D})$, where $\mathfrak{m}$ denotes the maximal ideal of Λ. This would mean that the Pontryagin dual $\mathcal{X}$ of $S_\mathcal{L}(K,\mathcal{D})$ has no nonzero, finite Λ-submodules. For if $\mathcal{Z}$ is the maximal finite Λ-submodule of $\mathcal{X}$, then $\mathcal{Z}[\mathfrak{m}] = \mathcal{X}[\mathfrak{m}]$ is the Pontryagin dual of $S_\mathcal{L}(K,\mathcal{D})/\mathfrak{m}S_\mathcal{L}(K,\mathcal{D})$. This is trivial if and only if $\mathcal{Z}$ itself is trivial.

Assume that $\mathrm{SUR}(\mathcal{D},\mathcal{L})$ is satisfied and that $H^1(K_\Sigma/K,\mathcal{D})$ is almost Λ-divisible. One sees easily that if $\mathcal{Z} \neq 0$, then $\mathcal{Z}[\Pi] \neq 0$ for all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. As a consequence of the snake lemma sequence, if one can show that α_Π is surjective for infinitely many Π's in $\mathrm{Spec}_{ht=1}(\Lambda)$, then it would follow that $\mathfrak{m}S_\mathcal{L}(K,\mathcal{D}) = S_\mathcal{L}(K,\mathcal{D})$. This observation is especially useful if Λ has Krull dimension 2. In that case, it follows that $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible if and only if α_Π is surjective for infinitely many Π's in $\mathrm{Spec}_{ht=1}(\Lambda)$. $\Diamond$

3.3. Behavior of the corank hypothesis under specialization. We can now complete the discussion in section 2.4. We want to justify the following equivalence.

- $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ *is true if and only if* $\mathrm{CRK}(\mathcal{D}[\Pi], \mathcal{L}_\Pi)$ *is true for almost all* $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

According to remark 2.1.3 in [Gr4], $\mathrm{coker}(\phi)$ is Λ-cotorsion if and only if $\mathrm{coker}\big(\alpha_\Pi)$ is (Λ/Π)-cotorsion for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. If we assume that $\mathcal{L}$ is almost Λ-divisible, then we have the isomorphism (4) for almost all Π's. Since Λ/Π is a finitely-generated Λ_Π-module, it follows that $\mathrm{coker}(\phi)$ is Λ-cotorsion if and only if $\mathrm{coker}\big(\phi_\Pi\big)$ is Λ_Π-cotorsion for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$, which is the stated equivalence.

The assumption that $\mathcal{L}$ is almost Λ-divisible is not needed. Suppose that $\Pi = (\pi)$ is an arbitrary element of $\mathrm{Spec}_{ht=1}(\Lambda)$. Referring to the discussion in section 3.2, we have an injective map

$$\mathrm{coker}\big(\phi_\Pi\big) \longrightarrow \mathrm{coker}\big(\alpha_\Pi\big) \quad . \tag{5}$$

induced by q_Π. Furthermore, the cokernel of (5) is isomorphic to $\mathrm{coker}\big(q_\Pi\big)$. The stated equivalence will follow if we show that $\mathrm{coker}\big(q_\Pi\big)$ is (Λ/Π)-cotorsion for almost all Π's. Using the notation from section 3.2, we have $\mathrm{coker}\big(q_\Pi\big) = \mathrm{coker}\big(c_\Pi\big)$. We then obtain another injective map

$$\mathrm{coker}\big(q_\Pi\big) \longrightarrow L(K, \mathcal{D})/\pi L(K, \mathcal{D}) \quad .$$

Thus, it suffices to show that $L(K, \mathcal{D})\big/\Pi L(K, \mathcal{D})$ is (Λ/Π)-cotorsion for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

In general, suppose that $\mathcal{A}$ is a discrete, cofinitely-generated Λ-module and that $\mathcal{X}$ is the Pontryagin dual of $\mathcal{A}$. Thus, we have a perfect pairing $\mathcal{A} \times \mathcal{X} \to \mathbf{Q}_p/\mathbf{Z}_p$. Let $\mathcal{Z}$ denote the maximal pseudo-null Λ-submodule of $\mathcal{X}$ and let $\mathcal{B}$ denote the orthogonal complement of $\mathcal{Z}$ under that pairing. Thus, $\mathcal{B} \subseteq \mathcal{A}$. Let $\mathcal{C} = \mathcal{A}/\mathcal{B}$. The Pontryagin duals of $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{X}/\mathcal{Z}$ and $\mathcal{Z}$, respectively. It follows that $\mathcal{B}$ is the maximal almost Λ-divisible Λ-submodule of $\mathcal{A}$. Furthermore, by definition, $\mathcal{Z}$ is annihilated by a nonzero element of Λ relatively prime to π, and so $\mathcal{Z}[\Pi]$ is a torsion (Λ/Π)-module. Thus, $\mathcal{C}\big/\Pi\mathcal{C}$ is (Λ/Π)-cotorsion. If we choose Π so that $\Pi\mathcal{B} = \mathcal{B}$, it follows that $\mathcal{A}/\Pi\mathcal{A} \cong \mathcal{C}/\Pi\mathcal{C}$. Applying these considerations to $\mathcal{A} = L(K, \mathcal{D})$, we see that $L(K, \mathcal{D})\big/\Pi L(K, \mathcal{D})$ is indeed (Λ/Π)-cotorsion for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

3.4. The case where $\phi_{\mathcal{L}}$ is not surjective. We assume in this section that $\mathcal{D}$, $H^1(K_\Sigma/K,\mathcal{D})$, and the specification $\mathcal{L}$ are almost Λ-divisible, but not that $\mathrm{SUR}(\mathcal{D},\mathcal{L})$ holds. In the exact sequence (3), we can simply replace $Q_{\mathcal{L}}(K,\mathcal{D})$ by the image of $\phi=\phi_{\mathcal{L}}$, which we will denote by $Q'_{\mathcal{L}}(K,\mathcal{D})$. We then can consider the map

$$\alpha'_\Pi \;:\; H^1(K_\Sigma/K,\mathcal{D})[\Pi] \;\longrightarrow\; Q'_{\mathcal{L}}(K,\mathcal{D})[\Pi] \;\;.$$

Applying the snake lemma as before, and using the assumption that $H^1(K_\Sigma/K,\mathcal{D})$ is almost Λ-divisible, we see that $S_{\mathcal{L}}(K,\mathcal{D})$ is almost Λ-divisible if and only if α'_Π is surjective for almost all $\Pi\in\mathrm{Spec}_{ht=1}(\Lambda)$.

We have an exact sequence

$$0\longrightarrow Q'_{\mathcal{L}}(K,\mathcal{D})\longrightarrow Q_{\mathcal{L}}(K,\mathcal{D})\longrightarrow \mathrm{coker}(\phi)\longrightarrow 0 \quad . \tag{6}$$

The kernel of the natural map $\xi_\Pi : Q_{\mathcal{L}}(K,\mathcal{D})[\Pi]\to\mathrm{coker}(\phi)[\Pi]$ is $Q'_{\mathcal{L}}(K,\mathcal{D})[\Pi]$ which clearly contains $\mathrm{im}(\alpha_\Pi)=\mathrm{im}(\alpha'_\Pi)$. We then obtain a map

$$\tilde{\xi}_\Pi :\; \mathrm{coker}(\alpha_\Pi) \;\longrightarrow\; \mathrm{coker}(\phi)[\Pi]$$

whose kernel is $\mathrm{coker}(\alpha'_\Pi)$. Thus, α'_Π is surjective if and only if $\ker(\tilde{\xi}_\Pi)$ is trivial.

Choose $\Pi\in\mathrm{Spec}_{ht=1}(\Lambda)$ so that the assumptions in lemma 3.2.1 are satisfied. Then q_Π induces the isomorphism (4) and hence determines an isomorphism from $\ker\big(\tilde{\xi}_\Pi\big)$ to a certain subgroup of $\mathrm{coker}(\phi_\Pi)$, namely the subgroup

$$q_\Pi^{-1}\big(\ker(\xi_\Pi)\big)\big/\mathrm{im}(\phi_\Pi) \;=\; \ker\big(\xi_\Pi\circ q_\Pi\big)\big/\mathrm{im}(\phi_\Pi) \;\;. \tag{7}$$

Note that $\xi_\Pi\circ q_\Pi$ is a map from $Q_{\mathcal{L}_\Pi}(K,\mathcal{D}[\Pi])$ to $\mathrm{coker}(\phi)[\Pi]$ and induces a map from $\mathrm{coker}(\phi_\Pi)$ to $\mathrm{coker}(\phi)[\Pi]$ whose kernel is (7). We can conclude that α'_Π is surjective if and only if the map

$$\mathrm{coker}(\phi_\Pi) \;\longrightarrow\; \mathrm{coker}(\phi)[\Pi] \tag{8}$$

is injective.

Consequently, $S_{\mathcal{L}}(K,\mathcal{D})$ is almost Λ-divisible if and only if (8) is injective for almost all $\Pi\in\mathrm{Spec}_{ht=1}(\Lambda)$. Note that (8) is surjective for almost all Π's. To see this, note that $Q'_{\mathcal{L}}(K,\mathcal{D})$ is a quotient of $H^1(K_\Sigma/K,\mathcal{D})$ and hence is almost Λ-divisible. Applying the snake lemma to (6), it follows that ξ_Π is surjective for almost all $\Pi\in\mathrm{Spec}_{ht=1}(\Lambda)$. The same is true for the map q_Π, hence for $\xi_\Pi\circ q_\Pi$, and therefore for the map (8) .

Examples exist where ϕ is not surjective, but α'_Π is surjective for almost all $\Pi\in\mathrm{Spec}_{ht=1}(\Lambda)$. This is discussed in the next section. The type of example considered there involves choosing a suitable $\mathbf{Z}_p$-extension K_∞ of K and a suitable β in the maximal ideal of Λ to define a

Galois module $\mathcal{T}^*$ of Λ-rank 1. The Galois module $\mathcal{D}$ is then $\mathrm{Hom}(\mathcal{T}^*, \mu_{p^\infty})$. When we take $\Lambda = \mathbf{Z}_p[[T]]$, both groups in (8) are finite for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Hence injectivity follows from surjectivity by simply showing that their orders are equal.

3.5. A special case. We discuss a way to verify that (8) is injective in a very special situation. A specific example will be given at the end of section 4.4. In general, suppose that $\mathcal{D}$ satisfies the assumptions in proposition 2.6.2 and that $\mathrm{LEO}(\mathcal{D})$ and $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ are satisfied. Then $Ш^1(K, \Sigma, \mathcal{T}^*) = 0$. Suppose also that

$$L(K_v, \mathcal{D}) \subseteq H^1(K_v, \mathcal{D})_{\Lambda-div} \tag{9}$$

for all $v \in \Sigma$. Under these assumptions, we have

$$\widehat{\mathrm{coker}(\phi)} \;\cong\; H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda-tors} \;\cong\; H^0(K, \mathcal{T}^*/\theta\mathcal{T}^*)$$

as Λ-modules, where $\theta \in \Lambda$ is any nonzero annihilator for $H^1(K_\Sigma/K, \mathcal{T}^*)_{\Lambda-tors}$. The first isomorphism follows from propositions 2.3.1 and 3.1.1 in [Gr5]. The second follows from proposition 2.2.2 in that paper.

Now assume also that $\mathcal{D}$ is a cofree Λ-module of corank 1. Then $\mathcal{T}^*$ is a free Λ-module of rank 1. Suppose that the action of G_K on $\mathcal{T}^*$ factors through $\Gamma = \mathrm{Gal}(K_\infty/K)$, where K_∞ is a $\mathbf{Z}_p$-extension of K. Hence the image of G_K in $\Lambda^\times$ is generated topologically by an element $1+\beta$, where $\beta \in \mathfrak{m}$. We assume that $p \nmid \beta$. Note that $1+\beta$ has infinite order. We can choose θ (as above) so that $\beta|\theta$. We then have

$$\widehat{\mathrm{coker}(\phi)} \;\cong\; (\beta^{-1}\theta)/\theta \;\cong\; \Lambda/(\beta) \tag{10}$$

as Λ-modules. Let $B = (\beta)$. Therefore, $\mathrm{coker}(\phi)[\Pi]$ is isomorphic to the Pontryagin dual of $\Lambda/(B+\Pi)$ as discrete Λ-modules.

In addition to the above assumptions, let us now assume that $\Lambda \cong \mathbf{Z}_p[[T]]$. Then Λ has Krull dimension 2 and Λ/B is a free $\mathbf{Z}_p$-module of some rank. Furthermore, if $\Pi \neq (p)$, then Λ/Π is a finite integral extension of $\mathbf{Z}_p$ and is free as a $\mathbf{Z}_p$-module. Note that $\mathcal{D}[\Pi]$ is $\mathbf{Z}_p$-cofree and hence $\mathbf{Z}_p$-divisible. If $B \not\subset \Pi$, then $\Lambda/(B+\Pi)$ is finite. Since the map (8) is surjective, injectivity will follow if one can verify that $\mathrm{coker}(\phi_\Pi)$ has the same order as $\Lambda/(B+\Pi)$.

We add one more assumption. For each $v \in \Sigma$, let Γ_v be the decomposition subgroup of Γ for v. We will assume that Γ_v is nontrivial for all $v \in \Sigma$. Thus, $[\Gamma : \Gamma_v]$ is finite. A topological generator for Γ_v acts as multiplication by $1+\beta_v$, where $\beta_v \in \mathfrak{m}$. Note that $1+\beta_v = (1+\beta)^a$, where $a \in \mathbf{Z}_p$ and $a \neq 0$. Since $p \nmid \beta$, it follows that $p \nmid \beta_v$. Let $B_v = (\beta_v)$. Then $H^0(K_v, \mathcal{D}) = \mathcal{D}[B_v]$ is cofree as a Λ/B_v-module and hence is almost divisible as a

Λ-module. It follows that $h_{\Pi,v}$ is an isomorphism for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$ and all $v \in \Sigma$. Also, as we show in the lemma below, $H^1(K_v, \mathcal{D})_{\Lambda-div}$ is Λ-cofree. Consequently, $H^1(K_v, \mathcal{D})_{\Lambda-div}[\Pi]$ is $\mathbf{Z}_p$-cofree (e.g., $\mathbf{Z}_p$-divisible) if $p \notin \Pi$. Assuming that $h_{\Pi,v}$ is injective, the same is true for its inverse image under the map $h_{\Pi,v}$. Consequently, for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$, the inclusion (9) holds when $\mathcal{D}$ is replaced by $\mathcal{D}[\Pi]$.

Note that if $p \notin \Pi$, then $\mathcal{D}[\Pi]$ is $\mathbf{Z}_p$-cofree of finite corank. It is just a divisible group. Furthermore, for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$, $\mathrm{LEO}(\mathcal{D}[\Pi])$ and $\mathrm{CRK}(\mathcal{D}[\Pi], \mathcal{L}_\Pi)$ are both satisfied. (See sections 2.4 and 3.3.) For such Π, $Ш^1(K, \Sigma, \mathcal{T}^*/\Pi\mathcal{T}^*)$ vanishes, $\mathrm{coker}(\phi_\Pi)$ is finite, and we can determine its order when the analogue of the inclusion (9) holds for $\mathcal{D}[\Pi]$. In what follows, we will denote $\mathcal{T}^*/\Pi\mathcal{T}^*$ more simply by $\mathcal{T}^*_\Pi$.

For Π as above, propositions 2.3.1 and 3.1.1 in [Gr5] imply that $\mathrm{coker}(\phi_\Pi)$ is isomorphic to the Pontryagin dual of $H^1(K_\Sigma/K, \mathcal{T}^*_\Pi)_{\mathbf{Z}_p-tors}$. Since this last group is finite, we can choose m sufficiently large so that p^m annihilates that group. If we assume that $\beta \notin \Pi$, then proposition 2.2.2 in [Gr5] implies that this last group is isomorphic to $H^0(K, \mathcal{T}^*_\Pi/p^m\mathcal{T}^*_\Pi)$. Hence, $\mathrm{coker}(\phi_\Pi)$ has the same order as the kernel of multiplication by β on the finite group $\mathcal{T}^*_\Pi/p^m\mathcal{T}^*_\Pi$. This is the same as the order of the cokernel of multiplication by β on that group, which is $\mathcal{T}^*_\Pi/(\beta, p^m)\mathcal{T}^*_\Pi$. By taking $m >> 0$, one can conclude that $\mathrm{coker}(\phi_\Pi)$ has the same order as $\mathcal{T}^*_\Pi/\beta\mathcal{T}^*_\Pi$ since that group is finite. Since $\mathcal{T}^*$ is free of rank 1 over Λ, it follows that $\mathrm{coker}(\phi_\Pi)$ has the same order as $\Lambda/(\Pi + B)$ for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$, which is indeed equal to the order of $\mathrm{coker}(\phi)[\Pi]$

To complete this discussion, we need the following lemma.

Lemma 3.5.1. *With the above assumptions, $H^1(K_v, \mathcal{D})_{\Lambda-div}$ is Λ-cofree.*

Proof. Let $c_v = \mathrm{corank}_\Lambda\big(H^1(K_v, \mathcal{D})\big)$. Of course, we also have $c_v = \mathrm{corank}_\Lambda\big(H^1(K_v, \mathcal{D})_{\Lambda-div}\big)$. It suffices to show that $H^1(K_v, \mathcal{D})_{\Lambda-div}[\Pi]$ is (Λ/Π)-cofree of corank c_v for at least one prime ideal $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. For it would then follow by Nakayama's lemma that the Pontryagin dual of $H^1(K_v, \mathcal{D})_{\Lambda-div}$ can be generated by c_v elements as a Λ-module and hence must be a free Λ-module of rank c_v.

For almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$, the (Λ/Π)-coranks of $H^1(K_v, \mathcal{D})_{\Lambda-div}[\Pi]$ and $H^1(K_v, \mathcal{D})[\Pi]$ are both equal to c_v. This follows from remark 2.1.3 in [Gr4]. We will assume in this proof that Π is chosen in that way. To simplify the discussion, we will also assume that Π is chosen so that $\Lambda/\Pi \cong \mathbf{Z}_p$. Define

$$A_v = H^1(K_v, \mathcal{D})/H^1(K_v, \mathcal{D})_{\Lambda-div}, \qquad A_{\Pi,v} = H^1(K_v, \mathcal{D}[\Pi])/H^1(K_v, \mathcal{D}[\Pi])_{\mathbf{Z}_p-div}\ .$$

By Poitou-Tate duality, the Pontryagin dual of A_v is isomorphic to $H^1(K_v, \mathcal{T}^*)_{\Lambda-tors}$. Just as argued above in the global case, it follows that $A_v[\Pi]$ is finite and is isomorphic to the Pontryagin dual of $\Lambda/(B_v + \Pi)$ for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Furthermore, we have an isomorphism between $H^1(K_v, \mathcal{T}^*/\Pi\mathcal{T}^*)_{\mathbf{Z}_p-tors}$ and $\Lambda/(B_v+\Pi)$ for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. By Poitou-Tate duality again, the Pontryagin dual of $H^1(K_v, \mathcal{T}^*/\Pi\mathcal{T}^*)_{\mathbf{Z}_p-tors}$ is in turn isomorphic to $A_{\Pi,v}$. Thus, it follows that $A_v[\Pi]$ is finite and isomorphic to $A_{\Pi,v}$ for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. We assume that Π is chosen in this way.

Since Π is principal, a snake lemma argument gives us the following exact sequence.

$$0 \longrightarrow H^1(K_v, \mathcal{D})_{\Lambda-div}[\Pi] \longrightarrow H^1(K_v, \mathcal{D})[\Pi] \longrightarrow A_v[\Pi] \longrightarrow 0 \quad .$$

By definition, we also have the exact sequence

$$0 \longrightarrow H^1(K_v, \mathcal{D}[\Pi])_{\mathbf{Z}_p-div} \longrightarrow H^1(K_v, \mathcal{D}[\Pi]) \longrightarrow A_{\Pi,v} \longrightarrow 0 \quad .$$

Now the natural map $H^1(K_v, \mathcal{D}[\Pi]) \to H^1(K_v, \mathcal{D})[\Pi]$ is an isomorphism for almost all Π because $H^0(K_v, \mathcal{D}) = \mathcal{D}[\beta_v]$ is an almost divisible Λ-module. For such Π, it is clear that the image of $H^1(K_v, \mathcal{D}[\Pi])_{\mathbf{Z}_p-div}$ under that natural map is precisely the maximal $\mathbf{Z}_p$-divisible submodule of $H^1(K_v, \mathcal{D})[\Pi]$ and hence is contained in $H^1(K_v, \mathcal{D})_{\Lambda-div}[\Pi]$. The fact that $A_v[\Pi]$ and $A_{\Pi,v}$ have the same order implies that the natural map induces an isomorphism

$$H^1(K_v, \mathcal{D}[\Pi])_{\mathbf{Z}_p-div} \quad \longrightarrow \quad H^1(K_v, \mathcal{D})_{\Lambda-div}[\Pi]$$

of (Λ/Π)-modules. Since the $\mathbf{Z}_p$-coranks of each is c_v, it then follows that $H^1(K_v, \mathcal{D})_{\Lambda-div}[\Pi]$ is indeed (Λ/Π)-cofree of corank c_v. ■

Remark 3.5.2. Suppose that $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$ and that $A_v[\Pi]$ has positive (Λ/Π)-corank. Since A_v is a cofinitely generated, cotorsion Λ-module, this means that $\widehat{A_v}[\Pi]$ has positive (Λ/Π)-rank, and so the same is true for $H^1(K_v, \mathcal{T}^*)_{\Lambda-tors}[\Pi]$. Consequently, $H^0(K_v, \mathcal{T}^*/\Pi\mathcal{T}^*)$ has positive (Λ/Π)-rank. Now

$$\mathcal{T}^*/\Pi\mathcal{T}^* \;\cong\; \mathrm{Hom}(\mathcal{D}[\Pi], \mu_{p^\infty}) \quad .$$

If $\Pi \neq (p)$, it follows $A_v[\Pi]$ has positive (Λ/Π)-corank if and only if $\mathrm{Hom}_{G_{K_v}}(\mathcal{D}[\Pi], \mu_{p^\infty})$ is infinite.

Assume now that G_{K_v} acts on $\mathcal{D}[\Pi]$ through a finite quotient group. Since $p \nmid \beta_v$, one sees easily that $\Pi \neq (p)$. Note that $K_v(\mu_{p^\infty})/K_v$ is an infinite extension. Consequently, it follows that $A_v[\Pi]$ is finite. Furthermore, if J is a product of such prime ideals, then $A_v[J]$ is also finite. Therefore, if L_v is a Λ-submodule of $H^1(K_v, \mathcal{D})$ which is annihilated by such an ideal J and if L_v is divisible as a group, then we must have the inclusion $L_v \subseteq H^1(K_v, \mathcal{D})_{\Lambda-div}$.

4 Sufficient conditions for almost divisibility.

We will prove a rather general result in section 4.1. Section 4.2 discusses the verification of various hypotheses in that result. Section 4.3 will concern a special case (although still quite general) where several of the hypotheses are automatically satisfied.

4.1. The main theorem. We prove the following result.

Proposition 4.1.1. *Suppose that* $\mathrm{RFX}(\mathcal{D})$ *and* $\mathrm{LEO}(\mathcal{D})$ *are satisfied, that* $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ *is satisfied for all* v *in* Σ*, and that there exists a non-archimedean prime* $\eta \in \Sigma$ *such that* $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ *is satisfied. Suppose also that* $\mathcal{L}$ *is almost divisible, that* $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ *is satisfied, and also that at least one of the following additional assumptions is satisfied.*

(a) $\mathcal{D}[\mathfrak{m}]$ *has no subquotient isomorphic to* μ_p *for the action of* G_K*,*

(b) $\mathcal{D}$ *is a cofree* Λ*-module and* $\mathcal{D}[\mathfrak{m}]$ *has no quotient isomorphic to* μ_p *for the action of* G_K*,*

(c) There is a prime $\eta \in \Sigma$ *which satisfies* $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ *and such that* $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ *is coreflexive as a* Λ*-module.*

Then $S_{\mathcal{L}}(K, \mathcal{D})$ *is an almost divisible* Λ*-module.*

Proof. First of all, $\mathrm{RFX}(\mathcal{D})$, $\mathrm{LEO}(\mathcal{D})$, and the assumptions about $\mathrm{LOC}_v^{(1)}$ and $\mathrm{LOC}_v^{(2)}$ are sufficient to imply that $H^1(K_\Sigma/K, \mathcal{D})$ is an almost divisible Λ-module. This follows from proposition 2.6.1. Secondly, since $\mathrm{RFX}(\mathcal{D})$ holds, $\mathcal{D}$ is certainly Λ-divisible. We can apply proposition 2.6.3 to conclude that $\mathrm{SUR}(\mathcal{D}, \mathcal{L})$ is satisfied too.

Thus, as described in section 3.1, it suffices to show that the map

$$\alpha_\Pi : \; H^1(K_\Sigma/K, \mathcal{D})[\Pi] \;\longrightarrow\; Q_{\mathcal{L}}(K, \mathcal{D})[\Pi]$$

is surjective for almost all $\Pi = (\pi)$ in $\mathrm{Spec}_{ht=1}(\Lambda)$. In the rest of this proof, we will exclude finitely many Π's in $\mathrm{Spec}_{ht=1}(\Lambda)$ in each step, and altogether just finitely many. We will follow the approach outlined in section 3, reducing the question to studying $\mathrm{coker}(\phi_\Pi)$ and then applying proposition 2.6.3. We want to apply that proposition to $\mathcal{D}[\Pi]$ and so must verify the appropriate hypotheses. At each step, we consider just the Π's which have not been already excluded. As described in section 2, we regard various (Λ/Π)-modules as modules over a certain subring Λ_Π.

Since $\mathrm{RFX}(\mathcal{D})$ holds for $\mathcal{D}$, it follows that $\mathcal{D}[\Pi]$ is a divisible (Λ/Π)-module. Corollary 2.6.1 in [Gr4] justifies that assertion. Therefore, $\mathcal{D}[\Pi]$ is also divisible as a Λ_Π-module. Furthermore, the assumption $\mathrm{LEO}(\mathcal{D})$ means that $Ш^2(K, \Sigma, \mathcal{D})$ is Λ-cotorsion. Consequently,

$Ш^2(K, \Sigma, \mathcal{D})[\Pi]$ is a cotorsion (Λ/Π)-module for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. This follows from remark 2.1.3 in [Gr4]. The same is true for $Ш^2(K, \Sigma, \mathcal{D}[\Pi])$ according to lemma 4.1.1 in [Gr4]. Recall that Λ/Π is finitely-generated as a Λ_Π-module. It follows that $\mathrm{LEO}(\mathcal{D}[\Pi])$ holds for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$.

The fact that $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ is satisfied implies that $\mathrm{CRK}(\mathcal{D}[\Pi], \mathcal{L}_\Pi)$ is satisfied for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. This follows from section 3.4. Thus, we can assume from here on that $\mathrm{coker}(\phi_\Pi)$ is Λ_Π-cotorsion. Now we consider the additional assumptions. Each implies the corresponding assumption in proposition 2.6.3. Once we verify that, it will then follow that ϕ_Π is surjective for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Hence the same thing will be true for α_Π. This will prove that $S_\mathcal{L}(K, \mathcal{D})$ is indeed almost divisible as a Λ-module.

First assume that *(a)* is satisfied. Let $\mathfrak{m}_\Pi$ denote the maximal ideal of Λ_Π. Using proposition 3.8 in [Gr4], it follows that $\mathcal{D}[\Pi][\mathfrak{m}_\Pi]$ indeed has no subquotient isomorphic to μ_p. Now assume that *(b)* is satisfied. Then $\mathcal{D}[\Pi]$ is cofree as a (Λ/Π)-module. Since Π is principal, Λ/Π is a complete intersection. According to proposition 3.1.20 in [BH], it follows that Λ/Π is a Cohen-Macaulay domain. Proposition 2.2.11 in [BH] then implies that Λ/Π is a free Λ_Π-module. Hence $\mathcal{D}[\Pi]$ is cofree as a Λ_Π-module. Furthermore, $\mathcal{D}[\mathfrak{m}] = \mathcal{D}[\Pi][\mathfrak{m}]$ has no quotient isomorphic to μ_p for the action of G_K. Remark 3.2.2 in [Gr5] implies that the same thing is true for $\mathcal{D}[\Pi][\mathfrak{m}_\Pi]$. Thus, the assumption *(b)* in proposition 2.6.3 for the Λ_Π-module $\mathcal{D}[\Pi]$ is indeed satisfied.

Now assume that *(c)* is satisfied. As pointed out in section 2.4, $\mathrm{LOC}_\eta^{(1)}(\mathcal{D}[\Pi])$ is satisfied for almost all $\Pi \in \mathrm{Spec}_{ht=1}(\Lambda)$. Since $\mathcal{D}$ is Λ-divisible and $L(K_\eta, \mathcal{D})$ is almost Λ-divisible, we have

$$Q_{\mathcal{L}_\Pi}(K_\eta, \mathcal{D}[\Pi]) \cong Q_\mathcal{L}(K_\eta, \mathcal{D})[\Pi]$$

for almost all Π's. It suffices to have $L(K_\eta, \mathcal{D})$ divisible by π. The assumption that $Q_\mathcal{L}(K_\eta, \mathcal{D})$ is a coreflexive Λ-module then implies that $Q_{\mathcal{L}_\Pi}(K_\eta, \mathcal{D}[\Pi])$ is (Λ/Π)-divisible, and hence Λ_Π-divisible, which is the only assumption in proposition 2.6.3*(c)* left to verify. ■

4.2. Non-primitive Selmer groups. Suppose that Σ_0 is a subset of Σ consisting of non-archimedean primes. Consider the map

$$\phi_{\mathcal{L},\Sigma_0} \ : \ H^1(K_\Sigma/K, \mathcal{D}) \longrightarrow \prod_{v \in \Sigma - \Sigma_0} Q_\mathcal{L}(K_v, \mathcal{D}) \quad .$$

We denote the kernel of $\phi_{\mathcal{L},\Sigma_0}$ by $S_\mathcal{L}^{\Sigma_0}(K, \mathcal{D})$. We refer to this group as the non-primitive Selmer group corresponding to the specification $\mathcal{L}$ and the set Σ_0. It is defined just as $S_\mathcal{L}(K, \mathcal{D})$, but one omits the local conditions for the specification $\mathcal{L}$ corresponding to the primes $v \in \Sigma_0$. Of course, we have the obvious inclusion $S_{\mathcal{L}}(K, \mathcal{D}) \subseteq S_{\mathcal{L}}^{\Sigma_0}(K, \mathcal{D})$ and the corresponding quotient $S_{\mathcal{L}}^{\Sigma_0}(K, \mathcal{D})/S_{\mathcal{L}}(K, \mathcal{D})$ is isomorphic to a Λ-submodule of $\prod_{v \in \Sigma_0} Q_{\mathcal{L}}(K_v, \mathcal{D})$. In effect, $S_{\mathcal{L}}^{\Sigma_0}(K, \mathcal{D})$ is the Selmer group corresponding to a new specification $\mathcal{L}'$, where we simply replace $L(K_v, \mathcal{D})$ by $L'(K_v, \mathcal{D}) = H^1(K_v, \mathcal{D})$ for all $v \in \Sigma_0$. Thus, we now have $Q_{\mathcal{L}'}(K_v, \mathcal{D}) = 0$ for $v \in \Sigma_0$.

If we assume that that $\mathrm{SUR}(\mathcal{D}, \mathcal{L})$ is satisfied, then obviously follows that $\mathrm{SUR}(\mathcal{D}, \mathcal{L}')$ is satisfied. Furthermore, we have

$$S_{\mathcal{L}}^{\Sigma_0}(K, \mathcal{D})/S_{\mathcal{L}}(K, \mathcal{D}) \;\cong\; \prod_{v \in \Sigma_0} Q_{\mathcal{L}}(K_v, \mathcal{D}) \;\; .$$

In general, $\mathrm{coker}\big(\phi_{\mathcal{L}'}\big)$ is clearly a quotient of $\mathrm{coker}\big(\phi_{\mathcal{L}}\big)$, and hence if we assume that $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ is satisfied, then so is $\mathrm{CRK}(\mathcal{D}, \mathcal{L}')$. The following proposition then follows immediately from proposition 4.1.1*(c)*.

Proposition 4.2.1 *Suppose that* $\mathrm{RFX}(\mathcal{D})$ *and* $\mathrm{LEO}(\mathcal{D})$ *are satisfied, that* $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ *is satisfied for all* v *in* Σ*, and that there exists a non-archimedean prime* $\eta \in \Sigma_0$ *such that* $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ *is satisfied. Suppose also that* $\mathcal{L}$ *is almost divisible and that* $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ *is satisfied. Then* $S_{\mathcal{L}}^{\Sigma_0}(K, \mathcal{D})$ *is an almost divisible* Λ*-module.*

Remark 4.2.2. Suppose that η is a non-archimedean prime not dividing p. Regarding $\mathcal{D}[\mathfrak{m}]$ as an $\mathbf{F}_p$-representation space for G_{K_η}, suppose that it has no subquotients isomorphic to μ_p or to $\mathbf{Z}/p\mathbf{Z}$ (with trivial action of G_{K_η}). According to proposition 3.1 in [Gr4], the G_{K_η}-module $\mathcal{D}[\mathfrak{m}^t]$ has the same property for all $t \geq 1$. The local duality theorems imply that $H^0(K_\eta, \mathcal{D}[\mathfrak{m}^t])$ and $H^2(K_\eta, \mathcal{D}[\mathfrak{m}^t])$ both vanish, and therefore that $H^1(K_\eta, \mathcal{D}[\mathfrak{m}^t]) = 0$. It follows that $H^1(K_\eta, \mathcal{D}) = 0$. If we let $\Sigma_0 = \{\eta\}$, then we have $S_{\mathcal{L}}^{\Sigma_0}(K, \mathcal{D}) = S_{\mathcal{L}}(K, \mathcal{D})$. The hypothesis $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ is also satisfied. Consequently, if the other assumptions in proposition 4.2.1 are satisfied, it follows that $S_{\mathcal{L}}(K, \mathcal{D})$ is almost divisible as a Λ-module. Alternatively, in this case, $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ vanishes and so is certainly coreflexive, making assumption *(c)* in proposition 4.1.1 satisfied.

4.3. Verifying the hypotheses. We will discuss the various hypotheses in proposition 4.1.1. Some of them are already needed for propositions 2.6.1 and 2.6.3, and we may simply refer to discussions in [Gr4] and [Gr5]. We have nothing additional to say about $\mathrm{RFX}(\mathcal{D})$. If $\mathcal{D}$ is R-cofree, then that hypothesis is just that R is a reflexive ring.

The local hypotheses. There is a discussion of the verification of $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ and $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ in section 5, part F of [Gr4]. Most commonly, $\mathrm{LOC}_v^{(1)}(\mathcal{D})$ is satisfied for all non-archimedean primes $v \in \Sigma$ simply because $H^0(K_v, \mathcal{T}^*) = 0$ for those v's. That is a rather mild condition, although we mention one kind of example in section 4.4 where it may fail to be satisfied. Such examples were one motivation for introducing $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ as a hypothesis in [Gr4]. Another motivation is that for archimedean primes, $H^0(K_v, \mathcal{T}^*)$ is often nontrivial, but $\mathrm{LOC}_v^{(2)}(\mathcal{D})$ may still be satisfied. The archimedean primes are only an issue when $p = 2$.

The hypotheses CRK($\mathcal{D}, \mathcal{L}$) and LEO($\mathcal{D}$). As mentioned before, a discussion of LEO($\mathcal{D}$) can be found in section 6, part D of [Gr4]. It is called hypothesis L there. Of course, the validity of CRK($\mathcal{D}, \mathcal{L}$) is related to the choice of the specification $\mathcal{L}$. We will discuss one rather natural way of choosing a specification below. Let $c_{\mathcal{L}}(K, \mathcal{D})$ denote the Λ-corank of the cokernel of $\phi_{\mathcal{L}}$. Thus, CRK($\mathcal{D}, \mathcal{L}$) means that $c_{\mathcal{L}}(K, \mathcal{D}) = 0$. As discussed in the introduction to [Gr5], one has an equation

$$s_{\mathcal{L}}(K, \mathcal{D}) \ = \ b_1(K, \mathcal{D}) - q_{\mathcal{L}}(K, \mathcal{D}) + c_{\mathcal{L}}(K, \mathcal{D}) + \mathrm{corank}_{\Lambda}\big(\text{Ш}^2(K, \Sigma, \mathcal{D})\big) \ ,$$

where $s_{\mathcal{L}}(K, \mathcal{D})$ and $q_{\mathcal{L}}(K, \mathcal{D})$ are the Λ-coranks of $S_{\mathcal{L}}(K, \mathcal{D})$ and $Q_{\mathcal{L}}(K, \mathcal{D})$, respectively. The integer $b_1(K, \mathcal{D})$ is defined just in terms of the Euler-Poincaré characteristic for $\mathcal{D}$ and the Λ-coranks of some local Galois cohomology groups, and does not depend on $\mathcal{L}$. It occurs in proposition 4.3 in [Gr4]. One then has a lower bound

$$s_{\mathcal{L}}(K, \mathcal{D}) \ \geq \ b_1(K, \mathcal{D}) - q_{\mathcal{L}}(K, \mathcal{D})$$

and equality means that both CRK($\mathcal{D}, \mathcal{L}$) and LEO($\mathcal{D}$) are satisfied. The simplest case is where $\mathcal{L}$ is chosen so that $q_{\mathcal{L}}(K, \mathcal{D}) = b_1(K, \mathcal{D})$. In this case, the equality means that $S_{\mathcal{L}}(K, \mathcal{D})$ is a cotorsion Λ-module.

The additional assumptions in proposition *4.1.1*. Remark 3.2.2 in [Gr5] discusses the additional assumptions *(a)* and *(b)*. It includes some observations when $\mathcal{D}$ arises from an n-dimensional representation ρ of $\mathrm{Gal}(K_\Sigma/K)$ over a ring R, as in the introduction. One observation is that if $n \geq 2$ and if the residual representation $\widetilde{\rho}$ is irreducible over the finite field $R/\mathfrak{M}$, then hypothesis *(a)* is satisfied. The residual representation gives the action of $\mathrm{Gal}(K_\Sigma/K)$ on $\mathcal{D}[\mathfrak{M}]$. Another observation in that remark is that $\mathcal{D}[\mathfrak{m}]$ has a quotient isomorphic to μ_p if and only if $\mathcal{D}[\mathfrak{M}]$ has such a quotient.

We now discuss hypothesis *(c)*. This will be useful if $\mathcal{D}[\mathfrak{m}]$ has a quotient or subquotient isomorphic to μ_p for the action of G_K. We will assume that η is a non-archimedean prime in Σ and that $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ is satisfied. The issue is the coreflexivity of $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ as a Λ-module.

Let us now make the following two assumptions: *(i)* $H^1(K_\eta, \mathcal{D})$ is Λ-coreflexive, *(ii)* $L(K_\eta, \mathcal{D})$ is almost Λ-divisible. The coreflexivity of the discrete Λ-module $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ then follows easily. To see this, suppose that $\mathcal{A}$ is a cofinitely generated, coreflexive, discrete Λ-module and that $\mathcal{B}$ is an almost divisible Λ-submodule of $\mathcal{A}$. Let $\mathcal{X}$ be the Pontryagin dual of $\mathcal{A}$ and let $\mathcal{Y}$ be the orthogonal complement of $\mathcal{B}$ under the perfect pairing $\mathcal{A} \times \mathcal{X} \to \mathbf{Q}_p/\mathbf{Z}_p$. Then $\mathcal{X}$ is a finitely-generated, reflexive Λ-module. Furthermore, $\mathcal{X}/\mathcal{Y}$ is the Pontryagin dual of $\mathcal{B}$ and hence has no nonzero pseudo-null Λ-submodules. However, the reflexive hull $\widetilde{\mathcal{Y}}$ of $\mathcal{Y}$ must be contained in $\mathcal{X}$ and the quotient $\widetilde{\mathcal{Y}}/\mathcal{Y}$ is a pseudo-null Λ-module, and so must be zero. It follows that $\mathcal{Y}$ is reflexive as a Λ-module and hence that its Pontryagin dual $\mathcal{A}/\mathcal{B}$ is a coreflexive Λ-module.

Section 5, part D, of [Gr4] gives some sufficient conditions for $H^1(K_\eta, \mathcal{D})$ to be coreflexive. One condition requires the assumption that μ_p is not a quotient of $\mathcal{D}[\mathfrak{m}]$ as a G_{K_η}-module. However, that assumption clearly implies assumption *(a)* in proposition 4.1.1. Another more subtle sufficient condition is given in proposition 5.9 in [Gr4]. It involves $\mathcal{T}^* \otimes_\Lambda \widehat{\Lambda}$ which is denoted by $\mathcal{D}^*$ there. We are assuming that $H^0(K_\eta, \mathcal{T}^*) = 0$. Equivalently, that means that $\mathcal{D}^*(K_\eta) = H^0(K_\eta, \mathcal{D}^*)$ is Λ-cotorsion. Its Pontryagin dual $\widehat{\mathcal{D}^*(K_\eta)}$ is a torsion Λ-module. The result from [Gr4] is that if $\mathcal{D}$ is Λ-cofree and if every associated prime ideal for the torsion Λ-module $\widehat{\mathcal{D}^*(K_\eta)}$ has height at least 3, then $H^1(K_\eta, \mathcal{D})$ is coreflexive as a Λ-module. Some interesting cases where this criterion is satisfied will be discussed in [Gr6].

Even if $H^1(K_\eta, \mathcal{D})$ fails to be coreflexive, it is still possible for the quotient Λ-module $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ to be coreflexive. Consider the following natural way to specify a choice of $L(K_\eta, \mathcal{D})$. Suppose that $\mathcal{C}_\eta$ is a G_{K_η}-invariant Λ-submodule of $\mathcal{D}$ and that $H^2(K_\eta, \mathcal{C}_\eta)$ vanishes. Then we can define

$$L(K_\eta, \mathcal{D}) \;=\; \mathrm{im}\big(\; H^1(K_\eta, \mathcal{C}_\eta) \;\longrightarrow\; H^1(K_\eta, \mathcal{D}) \;\big) \quad .$$

Let $\mathcal{E}_\eta = \mathcal{D}/\mathcal{C}_\eta$. The map $H^1(K_\eta, \mathcal{D}) \to H^1(K_\eta, \mathcal{E}_\eta)$ is surjective and its kernel is $L(K_\eta, \mathcal{D})$. If $\eta \nmid p$, then one can take $\mathcal{C}_\eta = 0$ and hence $L(K_\eta, \mathcal{D}) = 0$. This is often a useful choice. If $\eta | p$, then one often will make a nontrivial choice of $\mathcal{C}_\eta$. This kind of definition occurs in [Gr2] for primes above p when a Galois representation ρ satisfies something we called a "Panchiskin condition." (See section 4 in [Gr2].) Under the stated assumptions, we have

$$Q_{\mathcal{L}}(K_\eta, \mathcal{D}) \;\cong\; H^1(K_\eta, \mathcal{E}_\eta)$$

as Λ-modules. Propositions 5.8 and 5.9 from [Gr4] then give the following result.

Proposition 4.3.1. *In addition to the assumption that $H^2(K_\eta, \mathcal{C}_\eta) = 0$, suppose that either one of the following assumptions is satisfied.*

(i) $\mathcal{E}_\eta$ is Λ-coreflexive and $\mathcal{E}_\eta[\mathfrak{m}]$ has no subquotient isomorphic to μ_p as a G_{K_η}-module,

(ii) $\mathcal{E}_\eta$ *is* Λ*-cofree and every associated prime ideal for the* Λ*-module* $\widehat{\mathcal{E}_\eta^*(K_\eta)}$ *has height at least* 3.

Then the Λ*-module* $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ *is coreflexive.*

Concerning *(i)*, note that it may be satisfied even if assumption *(a)* in proposition 4.1.1 fails to be satisfied. One such situation will be mentioned in section 4.4.

We will also want $L(K_\eta, \mathcal{C}_\eta)$ to be almost Λ-divisible. The following result follows immediately from proposition 5.3 in [Gr4].

Proposition 4.3.2. *Assume that* $\mathcal{C}_\eta$ *is* Λ*-coreflexive and that* $H^2(K_\eta, \mathcal{C}_\eta) = 0$*. Then* $H^1(K_\eta, \mathcal{C}_\eta)$ *is almost* Λ*-divisible. Hence the image of* $H^1(K_\eta, \mathcal{C}_\eta)$ *in* $H^1(K_\eta, \mathcal{D})$ *is also almost* Λ*-divisible.*

4.4. The two classical results. Let p be an odd prime. Suppose that T is a free $\mathbf{Z}_p$-module of rank n which has an action of $\mathrm{Gal}(K_\Sigma/K)$. Thus, we have a continuous homomorphism $\mathrm{Gal}(K_\Sigma/K) \to \mathrm{Aut}_{\mathbf{Z}_p}(T)$. Suppose also that K_∞ is the cyclotomic $\mathbf{Z}_p$-extension of K and let $\Gamma = \mathrm{Gal}(K_\infty/K)$. Let $\Lambda = \mathbf{Z}_p[[\Gamma]]$ denote the completed group ring for Γ over $\mathbf{Z}_p$. Thus, Λ is isomorphic to a formal power series ring $\mathbf{Z}_p$ in one variable. In this situation, one can define a free Λ-module $\mathcal{T}$ of rank n together with a homomorphism $\rho : \mathrm{Gal}(K_\Sigma/K) \to \mathrm{Aut}_\Lambda(\mathcal{T})$. This is described in section 5 of [Gr5] in detail, where $\mathcal{T}$ is denoted by $T \otimes \kappa$, and also in [Gr2] where it is called the cyclotomic deformation of T. Here κ is the natural embedding of Γ into $\Lambda^\times$ and one thinks of $\mathcal{T}$ as the twist of T by the $\Lambda^\times$-valued character κ.

Just as in the introduction, taking $R = \Lambda$, one can define $\mathcal{D} = \mathcal{T} \otimes_\Lambda \widehat{\Lambda}$. This discrete, Λ-cofree $\mathrm{Gal}(K_\Sigma/K)$-module $\mathcal{D}$ is denoted by $D \otimes \kappa$ in [Gr5], where $D = T \otimes_{\mathbf{Z}_p} (\mathbf{Q}_p/\mathbf{Z}_p)$. We think of $\mathcal{D}$ as the $\mathrm{Gal}(K_\Sigma/K)$-module obtained from D by inducing from $\mathrm{Gal}(K_\Sigma/K_\infty)$ up to $\mathrm{Gal}(K_\Sigma/K)$. We have $D \cong \mathcal{D}[I]$ as $\mathrm{Gal}(K_\Sigma/K)$, where I denotes the augmentation ideal in Λ, Consequently, $D[p] \cong \mathcal{D}[\mathfrak{m}]$, where $\mathfrak{m}$ is the maximal ideal of Λ.

Many of our hypothesis are automatically satisfied. Obviously, $\mathrm{RFX}(\mathcal{D})$ is satisfied. Furthermore, lemma 5.2.2 in [Gr5] shows that $\mathrm{LOC}_\eta^{(1)}(\mathcal{D})$ is satisfied for all non-archimedean primes η in Σ. This is so because only the archimedean primes can split completely in K_∞/K. Since p is assumed to be odd, if η is archimedean, then $(\mathcal{T}^*)^{G_{K_\eta}}$ is a direct summand in $\mathcal{T}^*$ and hence $\mathrm{LOC}_\eta^{(2)}(\mathcal{D})$ is satisfied. It is reasonable to conjecture that $\mathrm{LEO}(\mathcal{D})$ is always satisfied. This is stated as conjecture 5.2.1 in [Gr5] and is equivalent to conjecture L stated in the introduction to [Gr4]. Section 5.2 in [Gr4] discusses its validity. It is proved in certain special cases. In the examples that we will discuss below, $\mathrm{LEO}(\mathcal{D})$ is indeed satisfied as well as $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$.

Consider the case where $T = T_p(E)$, the p-adic Tate module for an elliptic curve defined over K. We then have $T/pT \cong E[p]$. Let Σ be a finite set of primes of K including the primes dividing p, the infinite primes, and the primes where E has bad reduction. The properties of the Weil pairing $E[p] \times E[p] \to \mu_p$ show that assumption *(b)* is satisfied if and only if $E(K)$ has no element of order p. Assume that E has good, ordinary reduction at the primes of K lying over p. There is a natural choice of a specification $\mathcal{L}$ in this case because the Panchiskin condition is satisfied. See the discussion in section 4.3. One chooses $L(K_\eta, \mathcal{D}) = 0$ if $\eta \nmid p$. If $\eta | p$, let C_η denote the kernel of the reduction map $E[p^\infty] \to \overline{E}_\eta[p^\infty]$, where $\overline{E}_\eta$ is the reduction of E at η. Let $\mathcal{C}_\eta = C_\eta \otimes \kappa$. Then $\mathcal{E}_\eta = \overline{E}_\eta[p^\infty] \otimes \kappa$. Note that $\mathcal{L}$ is almost divisible.

The formulas in section 2.3 show that $\delta_\Lambda(K, \mathcal{D}) = [K : \mathbf{Q}]$, which is a lower bound on $\mathrm{corank}_\Lambda\big(H^1(K_\Sigma/K, \mathcal{D})\big)$. However, the local formulas show easily that $Q_{\mathcal{L}}(K_v, \mathcal{D})$ has Λ-corank 0 when $v \nmid p$ and Λ-corank $[K_v : \mathbf{Q}_p]$ when $v|p$. Therefore, $\mathrm{corank}_\Lambda\big(Q_{\mathcal{L}}(K, \mathcal{D})\big) = [K : \mathbf{Q}]$. Thus, if $S_{\mathcal{L}}(K, \mathcal{D})$ is Λ-cotorsion, then inequality (2) shows that both $\mathrm{LEO}(\mathcal{D})$ and $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ are satisfied.

The above discussion shows that if $E(K)$ has no element of order p and if $S_{\mathcal{L}}(K, \mathcal{D})$ is Λ-cotorsion, then proposition 4.1.1 implies that $S_{\mathcal{L}}(K, \mathcal{D})$ is an almost divisible Λ-module. The second classical result stated in the introduction follows from this because $\mathrm{Sel}_E(K_\infty)$ can be identified with the Selmer group attached to D over K_∞. However, proposition 3.2 in [Gr2] gives an isomorphism between that Selmer group and $S_{\mathcal{L}}(K, \mathcal{D})$ (with a Λ-module structure modified by the involution of Λ induced from $\gamma \to \gamma^{-1}$ for $\gamma \in \Gamma$).

Now suppose that K is totally real, that $T \cong \mathbf{Z}_p$, and that G_K acts on T by a totally odd character ψ. Since p is odd, the order of ψ divides $p-1$. Let Σ be a finite set of primes of K including the primes dividing p, the infinite primes, and the primes dividing the conductor of ψ. Define D and $\mathcal{D}$ as described above. Thus, $\mathcal{D}$ is Λ-cofree and has Λ-corank 1. We take the following specification $\mathcal{L}$:

$$L(K_v, \mathcal{D}) \;=\; \ker\big(H^1(K_v, \mathcal{D}) \to H^1(K_v^{unr}, \mathcal{D})\big)$$

for all $v \in \Sigma$. Here K_v^{unr} denotes the maximal unramified extension of K_v. Thus, $S_{\mathcal{L}}(K, \mathcal{D})$ consists of locally unramified cocycle classes in $H^1(K_\Sigma/K, \mathcal{D})$ (or equivalently, cocycle classes in $H^1(K, \mathcal{D})$ which are unramified at all primes v of K.). Just as in the elliptic curve case, one can identify $S_{\mathcal{L}}(K, \mathcal{D})$ (slightly modifying the Λ-module structure) with $S(K_\infty.D)$ (as defined in the introduction) and hence the Pontryagin dual of $S_{\mathcal{L}}(K, \mathcal{D})$ can be identified with $X^{(\psi)}$, where $X = \mathrm{Gal}(L_\infty/K_{\infty,\psi})$. Iwasawa proved that X is a finitely generated, torsion Λ-module. Hence, $S_{\mathcal{L}}(K, \mathcal{D})$ is a cofinitely generated, cotorsion Λ-module. As we explain below, $L(K_v, \mathcal{D})$ is Λ-cotorsion for all v. Furthermore, the formulas in section 2.3 show that the Λ-corank of $H^1(K_\Sigma/K, \mathcal{D})$ is at least $[K : \mathbf{Q}]$ and the Λ-corank of $Q_{\mathcal{L}}(K, \mathcal{D})$ is equal to $[K : \mathbf{Q}]$. The fact that $S_{\mathcal{L}}(K, \mathcal{D})$ has Λ-corank 0 implies that the Λ-corank of $H^1(K_\Sigma/K, \mathcal{D})$ is equal to $[K : \mathbf{Q}]$ and that $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ is satisfied. It also follows that $H^2(K_\Sigma/K, \mathcal{D})$ has

Λ-corank 0, and hence the same is true for $Ш^2(K_\infty, \Sigma, D)$. Thus, LEO($\mathcal{D}$) is satisfied. We now show that $\mathcal{L}$ is almost divisible.

Let $\mathcal{D}(K_v^{unr})$ denote $H^0(K_v^{unr}, \mathcal{D})$. The inflation-restriction sequence shows that

$$L(K_v, \mathcal{D}) \;\cong\; H^1\big(K_v^{unr}/K_v,\ \mathcal{D}(K_v^{unr})\big)$$

as Λ-modules. Let ψ_v be the restriction of ψ to the decomposition subgroup Δ_v of $\Delta = \mathrm{Gal}(K_\psi/K)$. Then ψ_v is a faithful character of Δ_v and has order dividing $p-1$. We can regard ψ_v as a character of G_{K_v} and it defines a faithful character of $\mathrm{Gal}(K_{v,\psi_v}/K_v)$ for a certain cyclic extension K_{v,ψ_v} of K_v. Let $K_{v,\infty}$ be the cyclotomic $\mathbf{Z}_p$-extension of K_v and let $\Gamma_v = \mathrm{Gal}(K_{v,\infty}/K)$. The action of G_{K_v} on $\mathcal{D}$ factors through $\mathrm{Gal}(K_{v,\psi_v}K_{v,\infty}/K_v)$ which is isomorphic to $\Delta_v \times \Gamma_v$, where we have identified Δ_v and Γ_v with subgroups of $\mathrm{Gal}(K_{v,\psi_v}K_{v,\infty}/K_v)$ in an obvious way. Note that Δ_v is a cyclic group of order dividing $p-1$. The inertia subgroup of Δ_v is also cyclic and a generator will act on $\mathcal{D}$ as multiplication by a root of unity ε_v of order dividing $p-1$.

If the restriction of ψ_v to $G_{K_v^{unr}}$ is nontrivial, then $\varepsilon_v \neq 1$ and hence $\mathcal{D}(K_v^{unr}) = 0$. It follows that $L(K_v, \mathcal{D}) = 0$ for such v. We assume now that ψ_v is unramified at v and hence $\varepsilon_v = 1$. The restriction map

$$H^1\big(K_v^{unr}/K_v,\ \mathcal{D}(K_v^{unr})\big) \;\longrightarrow\; H^1\big(K_v^{unr}/K_{v,\psi_v},\ \mathcal{D}(K_v^{unr})\big)^{\Delta_v}$$

is injective. Also, we have an isomorphism

$$(11) \qquad H^1\big(K_v^{unr}/K_{v,\psi_v},\ \mathcal{D}(K_v^{unr})\big)^{\Delta_v} \;\cong\; \mathrm{Hom}_{\Delta_v}\big(\Gamma_v,\ \mathcal{D}(K_v^{unr})\big/(\gamma_v - 1)\mathcal{D}(K_v^{unr})\big)\ .$$

The action of Δ_v on Γ_v (by conjugation) is trivial. On the other hand, Δ_v is cyclic and a generator δ_v acts on $\mathcal{D}$ as multiplication by a root of unity ζ_v of order dividing $p-1$. Hence, if ψ_v is nontrivial, then $\zeta_v \neq 1$ and $H^0\big(\Delta_v, \mathcal{D}(K_v^{unr})\big/(\gamma_v - 1)\mathcal{D}(K_v^{unr})\big)$ must vanish. It follows that the right side in (11) is trivial. Therefore, $L(K_v, \mathcal{D}) = 0$ in this case too.

We assume now that ψ_v is trivial and hence the action of G_{K_v} on $\mathcal{D}$ factors through $\mathrm{Gal}(K_{v,\infty}/K_v)$. If $v \nmid p$, then v is unramified in K_∞/K and hence $K_{v,\infty} \subset K_v^{unr}$. Thus, $\mathcal{D}(K_v^{unr}) = \mathcal{D}$. Furthermore, $\mathrm{Gal}(K_v^{unr}/K_v)$ contains a unique subgroup P isomorphic to $\mathbf{Z}_p$ and the restriction map $P \to \Gamma_v$ is an isomorphism. The action of P on $\mathcal{D}$ is through this isomorphism. Let γ_v be a topological generator for Γ_v. The restriction map

$$H^1(K_v^{unr}/K_v, \mathcal{D}) \;\longrightarrow\; H^1(P, \mathcal{D})$$

is injective. Also, $H^1(P, \mathcal{D}) \cong \mathcal{D}/(\gamma_v - 1)\mathcal{D}$ vanishes because $\gamma_v - 1$ acts on $\mathcal{D}$ as multiplication by a nonzero element of Λ and $\mathcal{D}$ is Λ-divisible. The above remarks show that $L(K_v, \mathcal{D}) = 0$ for all $v \nmid p$.

Now consider primes v of K lying over p. If ψ_v is nontrivial, then $L(K_v, \mathcal{D}) = 0$, as shown above. Assuming that ψ_v is trivial, the action of G_{K_v} on $\mathcal{D}$ factors through Γ_v. By definition, $\Gamma_v \subseteq \Gamma$ is identified with a subgroup of $\Lambda^\times = \mathbf{Z}_p[[\Gamma]]^\times$ in a canonical way, and one sees that γ_v acts on $\mathcal{D}$ as multiplication by $1 + \beta_v$, where $\beta_v \in \Lambda$ and $p \nmid \beta_v$. The inertia subgroup of Γ_v is topologically generated by $\gamma_v^{p^a}$ for some $a \geq 0$. Also, $\gamma_v^{p^a} - 1$ acts as multiplication by $\beta_v^{p^a} - 1$, an element of Λ which is not divisible by p. It follows that $\mathcal{D}(K_v^{unr}) = \mathcal{D}[\beta_v^{p^a} - 1]$ is a divisible group. It also follows that $\mathcal{D}(K_v^{unr})$ is a cotorsion Λ-module. Furthermore, as before, $H^1\big(K_v^{unr}/K_v, \mathcal{D}(K_v^{unr})\big)$ is isomorphic to a certain quotient of $\mathcal{D}(K_v^{unr})$. Hence $L(K_v, \mathcal{D})$ is cotorsion as a Λ-module and is a divisible group. Hence, its Pontryagin dual has no nonzero finite Λ-submodules. Thus, in this case, $L(K_v, \mathcal{D})$ may be nontrivial, but it is still almost divisible as a Λ-module.

Assume that $\psi \neq \omega$. Now $\mathcal{D}[\mathfrak{m}]$ is a 1-dimensional $\mathbf{F}_p$-vector space on which G_K acts by ψ. Hence, assumption *(a)* in proposition 4.1.1 is satisfied. It then follows that the Λ-module $S_{\mathcal{L}}(K, \mathcal{D})$ is indeed almost divisible, and hence Iwasawa's theorem is proved when $\psi \neq \omega$.

In the case $\psi = \omega$, then we are in the setting of section 3.5 and must use the results from sections 3.4 and 3.5. In this case, (10) shows that $\phi_{\mathcal{L}}$ is not surjective. We must show that $L(K_v, \mathcal{D}) \subseteq H^1(K_v, \mathcal{D})_{\Lambda-div}$ for all $v \in \Sigma$. Now, $L(K_v, \mathcal{D})$ is nontrivial only when $v|p$ and and ψ_v is trivial. But in that case, $L(K_v, \mathcal{D})$ is a quotient of $\mathcal{D}[\beta_v^{p^a} - 1]$ and is annihilated by $J = (\beta_v^{p^a} - 1)$. Now J is a product of prime ideals of height 1 which contain $\beta_v^{p^a} - 1$. Hence, G_{K_v} acts on $\mathcal{D}[\Pi]$ through a finite quotient group. Remark 3.5.2 then implies that $L(K_v, \mathcal{D}) \subseteq H^1(K_v, \mathcal{D})_{\Lambda-div}$. This is true for all $v \in \Sigma$. Consequently, $S_{\mathcal{L}}(K, \mathcal{D})$ is an almost divisible Λ-module.

4.5. Examples where almost divisibility fails. We consider two variants of the classical examples mentioned in section 4.4. We will follow the notation described there and the Selmer groups will be defined in exactly the same way. In one example, all the hypotheses in proposition 4.1.1 are satisfied, except that none of the additional assumptions *(a), (b)* or *(c)* hold. In another example, it is $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ which is not satisfied.

Let $p = 5$. Let E be the elliptic curve over $\mathbf{Q}$ of conductor 11 such that $E(\mathbf{Q}) = 0$. It is the second curve in Cremona's tables and has good, ordinary reduction at p. The curve E has an isogeny of degree p defined over $\mathbf{Q}$ whose kernel Φ is isomorphic to μ_p for the action of $\mathrm{G}_{\mathbf{Q}}$. Also, the action of $G_{\mathbf{Q}}$ on $E[p]/\Phi \cong \mathbf{Z}/p\mathbf{Z}$ is trivial. Let $K = \mathbf{Q}(\mu_p)$ and let $T = T_p(E)$ as in section 4.4. Note that $E(K)$ has a point of order p. A theorem of Kato, or a direct calculation, implies that $S_{\mathcal{L}}(K, \mathcal{D})$ is Λ-cotorsion, and hence $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ is satisfied. It is clear that $\mathcal{D}[\mathfrak{m}] = E[p]$ has a quotient $E[p]/\Phi$ isomorphic to μ_p for the action of G_K. Thus, assumptions *(a)* and *(b)* fail to hold. We take Σ to be the set of primes lying above ∞, p, or 11. Assumption *(c)* fails to hold too. For if η lies over 11, one finds that $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ is

Λ-cotorsion, but nontrivial, and hence cannot be coreflexive. If η lies over p, one finds that $Q_{\mathcal{L}}(K_\eta, \mathcal{D})$ is not Λ-divisible and hence is not coreflexive. In this example, $S_{\mathcal{L}}(K, \mathcal{D})$ can be identified with $\mathrm{Sel}_E(K_\infty)_p$ as Λ-modules (up to an involution of Λ). It is shown in [Gr3], pages 127-128, that $\mathrm{Sel}_E(K_\infty)_p$ has a direct summand as a Λ-module which is of order p. Hence, the Pontryagin dual of $\mathrm{Sel}_E(K_\infty)_p$ has a submodule isomorphic to $\Lambda/\mathfrak{m}$. And so, in this example, $S_{\mathcal{L}}(K, \mathcal{D})$ fails to be almost divisible as a Λ-module.

Let p be any odd prime. Suppose that K is a totally real number field, that $T \cong \mathbf{Z}_p$, and that G_K acts on T by a totally even character ψ. In this case, K_ψ is totally real. It is conjectured that $X = \mathrm{Gal}(L_\infty/K_{\infty,\psi})$ is finite. We refer the reader to [Gr7], pages 350, 351, for more discussion and references concerning this conjecture. There are many examples when $K = \mathbf{Q}$, ψ has order 2, and $p = 3$, where $X^{(\psi)}$ turns out to be finite, but nonzero. It would then follow that $S_{\mathcal{L}}(K, \mathcal{D})$ is finite and nonzero, and hence fails to be almost divisible as a Λ-module. The weak Leopoldt conjecture holds for the $\mathbf{Z}_p$-extension $K_{\infty,\psi}/K_\psi$ and this implies that the Λ-corank of $H^1(K_\Sigma/K, \mathcal{D})$ is zero. We refer the reader to page 344 in [Gr4] for an explanation. In contrast, the local formula in section 2.3 implies that $Q_{\mathcal{L}}(K, \mathcal{D})$ has positive Λ-corank. Therefore, $\mathrm{CRK}(\mathcal{D}, \mathcal{L})$ cannot be satisfied.

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