CONTROL OF Λ-ADIC MORDELL–WEIL GROUPS

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ABSTRACT. The (pro) Λ-MW group is a projective limit of Mordell–Weil groups over a number field k (made out of modular Jacobians) with an action of the Iwasawa algebra and the "big" Hecke algebra. We prove a control theorem of the ordinary part of the Λ-MW groups under mild assumptions. We have proven a similar control theorem for the dual completed inductive limit in [H15].

1. Introduction

Fix a prime p. This article concerns weight 2 cusp forms of level Np^r for $r > 0$ and $p \nmid N$, and for small primes $p = 2, 3$, they exists only when $N > 2$; thus, we may assume $Np^r \geq 4$. Then the open curve $Y_1(Np^r)$ (obtained from $X_1(Np^r)$ removing all cusps) gives the fine smooth moduli scheme classifying elliptic curves E with an embedding $\mu_{Np^r} \hookrightarrow E$. We applied in [H86b] and [H14] the techniques of $U(p)$-isomorphisms to Barsotti–Tate groups of modular Jacobian varieties of high p-power level (with the fixed prime-to-p level N). In this article, we apply the same techniques of $U(p)$-isomorphisms to the projective limit of Mordell–Weil groups of the Jacobians and see what we can say (see Section 3 for $U(p)$-isomorphisms). We study the (inductive limit of) Tate–Shafarevich groups of the Jacobians in another article [H16].

Let $X_r = X_1(Np^r)_{/\mathbb{Q}}$ be the compactified moduli of the classification problem of pairs (E, ϕ) of an elliptic curve E and an embedding $\phi : \mu_{Np^r} \hookrightarrow E[Np^r]$. Write $J_{r/\mathbb{Q}}$ for the Jacobian whose origin is given by the infinity cusp $\infty \in X_r(\mathbb{Q})$ of X_r. For a number field k, we consider the group of k-rational points $J_r(k)$. Put $\widehat{J}_r(k) := \varprojlim_n J_r(k)/p^n J_r(k)$ (as a compact p-profinite module). The Albanese functoriality of Jacobians (twisted by the Weil involutions) gives rise to a projective system $\{\widehat{J}_r(k)\}_r$ compatible with Hecke operators (see Section 6 for details of twisting), and we have

$$\widehat{J}_\infty(k) = \varprojlim_r \widehat{J}_r(k)$$

equipped with the projective limit compact topology. By Picard functoriality, we have an injective limit $J_\infty(k) = \varinjlim_r \widehat{J}_r(k)$ (with the injective limit of the compact topology of $\widehat{J}_r(k)$) and $J_\infty[p^\infty]_{/\mathbb{Q}} = \varinjlim_r J_r[p^\infty]_{/\mathbb{Q}}$ (the injective limit of the p-divisible Barsotti–Tate group). We define

$$\check{J}_\infty(k) = \varprojlim_n J_\infty(k)/p^n J_\infty(k).$$

An fppf sheaf $\mathcal{F}$ (over Spec(k)) is a presheaf functor from the fppf site over Spec(k) to the category of abelian groups satisfying the sheaf condition for an fppf covering $\{U_i\}$ of $T_{/k}$, that is, the exactness of

$$\text{(L)} \qquad 0 \to \mathcal{F}(T) \xrightarrow{\mathrm{Res}_{U_i/T}} \prod_i \mathcal{F}(U_i) \xrightarrow{\mathrm{Res}_{U_{ij}/U_i} - \mathrm{Res}_{U_{ij}/U_j}} \prod_{i,j} \mathcal{F}(U_{ij}),$$

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where $\mathrm{Res}_{U/V}$ indicates the restriction map relative to $U \to V$ and $U_{ij} := U_i \times_T U_j$. Since the category of fppf sheaves over $\mathbb{Q}$ (e.g., [EAI, §4.3.7]) is an abelian category (cf. [ECH, II.2.15]), if we apply a left exact functor (of the category of abelian groups into itself) to the value of a sheaf, it preserves the sheaf condition given by the left exactness (L). Thus projective limits and injective limits exist inside the category of fppf sheaves. We may thus regard

$$R \mapsto \widehat{J}_\infty(R) := \varprojlim_r (J_r(R) \otimes_{\mathbb{Z}} \mathbb{Z}_p) \text{ and } R \mapsto J_\infty(R)$$

as fppf sheaves over the fppf site over $\mathbb{Q}$ for an fppf extension $R_{/k}$, though we do not use this fact much (as we compute $\widehat{J}_\infty(k)$ as a limit of $\widehat{J}_s(k)$ not using sheaf properties of $\widehat{J}_\infty$). If one extends $\widehat{J}_s$ to the ind-category of fppf extensions, we no longer have projective limit expression. We have given detailed description of the value $\widehat{J}_s(R)$ in [H15, §2] and we will give a brief outline of this in Section 2 in the text. We can think of the sheaf endomorphism algebra $\mathrm{End}(J_{\infty/\mathbb{Q}})$ (in which we have Hecke operators $T(n)$ and $U(l)$ for $l|Np$).

The Hecke operator $U(p)$ acts on $J_r(k)$, and the p-adic limit $e = \lim_{n\to\infty} U(p)^{n!}$ is well defined on $\widehat{J}_r(k)$. As is well known (cf. [H86b] and [O99]; see an exposition on this in Section 6), $T(n)$, $U(l)$ and diamond operators are endomorphisms of the injective (resp. projective) systems $\{J_s(k)\}_s$ (resp. $\{\widehat{J}_s(k)\}_s$). The projective system comes from w-twisted Albanese functoriality for the Weil involution w (as we need to twist in order to make the system compatible with $U(p)$; see Section 6 for the twisting). The image of e is called the *ordinary* part. We attach as the superscript or the subscript "ord" to indicate the ordinary part. Since these $\mathbb{Z}_p$-modules have natural action of the Iwasawa algebra Λ through diamond operators, we call in particular the group $\widehat{J}_\infty(k)^{\mathrm{ord}}$ the pro Λ-MW group ("MW" stands for Mordell–Weil). We define the Λ-BT group $\mathcal{G}_{/\mathbb{Q}}$ by the ordinary part $J_\infty[p^\infty]^{\mathrm{ord}}_{/\mathbb{Q}}$ of $J_\infty[p^\infty]_{/\mathbb{Q}}$ whose detailed study is made in [H14, §4]. Though in [H14], we made an assumption that $p \geq 5$, as for the results over $\mathbb{Q}$ in [H14, §4], they are valid without any change for $p = 2, 3$ as verified in [GK13] for $p = 2$ (and the prime $p = 3$ can be treated in the same manner as in [H86a] or [H14, §4]). Thus we use control result over $\mathbb{Q}$ of $\mathcal{G}$ in this paper without assuming $p \geq 5$. Its Tate module $T\mathcal{G} := \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda^\vee, \mathcal{G})$ is a continuous $\Lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module under the profinite topology, where $M^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ (Pontryagin dual) for $\mathbb{Z}_p$-modules M. We define the big Hecke algebra $\mathbf{h} = \mathbf{h}(N)$ to be the Λ-subalgebra of $\mathrm{End}_\Lambda(T\mathcal{G})$ generated by Hecke operators $T(n)$ $(n = 1, 2, \dots)$. Then $\widehat{J}_\infty(k)^{\mathrm{ord}}$ and $\check{J}_\infty(k)^{\mathrm{ord}}$ are naturally continuous $\mathbf{h}$-modules. Take a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ and define the direct factors

$$\widehat{J}_s(k)^{\mathrm{ord}}_{\mathbb{T}} := \widehat{J}_s(k)^{\mathrm{ord}} \otimes_{\mathbf{h}} \mathbb{T} \ (s = 1, 2, \dots, \infty) \text{ and } T\mathcal{G}_{\mathbb{T}} := T\mathcal{G} \otimes_{\mathbf{h}} \mathbb{T}$$

of $\widehat{J}_\infty(k)^{\mathrm{ord}}$ and $T\mathcal{G}$, respectively. In this introduction, for simplicity, we assume that the component $\mathbb{T}$ cuts out $\widehat{J}_\infty(k)^{\mathrm{ord}}_{\mathbb{T}}$ from $\widehat{J}_\infty(k)^{\mathrm{ord}}$ a part with potentially good reduction modulo p (meaning that $\mathcal{G}_{\mathbb{T}}[\gamma^{p^s} - 1]$ extends to Λ-BT group over $\mathbb{Z}_p[\mu_{p^s}]$). This is to avoid technicality coming from potentially multiplicative reduction of factors of J_s outside $\widehat{J}_s(k)^{\mathrm{ord}}_{\mathbb{T}}$.

The maximal torsion-free part Γ of $\mathbb{Z}_p^\times$ (which is a p-profinite cyclic group) acts on these modules by the diamond operators. In other words, for modular curves X_r and $X_0(Np^r)$, we identify $\mathrm{Gal}(X_r/X_0(Np^r))$ with $(\mathbb{Z}/Np^r\mathbb{Z})^\times$, and Γ acts on J_r through its image in $\mathrm{Gal}(X_r/X_0(Np^r))$. Therefore the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\Gamma]] = \varprojlim_r \mathbb{Z}_p[\Gamma/\Gamma^{p^r}]$ acts on the pro Λ-MW group, the ind Λ-MW group, the Λ-BT group and its Tate module. Then $T\mathcal{G}$ is known to be free of finite rank over Λ [H86b], [GK13] and [H14, §4]. A prime $P \in \mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$ is called *arithmetic* of weight 2 if P factors through $\mathrm{Spec}(\mathbb{T} \otimes_\Lambda \mathbb{Z}_p[\Gamma/\Gamma^{p^r}])$ for some $r > 0$. Associated to P is a unique Hecke eigenform of weight 2 on $X_1(Np^r)$ for some $r > 0$. Write B_P for the Shimura's abelian quotient associated to f_P of the jacobian J_r. Let $\mathcal{A}_{\mathbb{T}}$ be the set of all principal arithmetic points of $\mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$ of weight 2 and put $\Omega_{\mathbb{T}} := \{P \in \mathcal{A}_{\mathbb{T}} | B_P \text{ has good reduction over } \mathbb{Z}_p[\mu_{p^\infty}]\}$. The word "principal" means, as a prime ideal of $\mathbb{T}$, it is generated by a single element, often written as α. In this article, we prove control results for the pro Λ-MW group $\widehat{J}_\infty(k)^{\mathrm{ord}}$ and study the control of the ind Λ-MW-groups $\check{J}_\infty(k)^{\mathrm{ord}}$ in the twin paper [H15, Theorem 6.5]. Take a topological generator $\gamma = 1 + p^\epsilon$ of Γ, and regard γ as a group element of $\Lambda = \mathbb{Z}_p[[\Gamma]]$, where $\epsilon = 1$ if $p > 2$ and $\epsilon = 2$ if $p = 2$. We use this definition of ϵ throughout the paper (and we assume that $r \geq \epsilon$ if the exponent $r-\epsilon$ shows up in a formula). We fix a finite set S of places of $\mathbb{Q}$ containing all places $v|Np$ and the archimedean place. Here is a simplified statement of our finial result:

Theorem. *If $\mathbb{T}$ is an integral domain, for almost all principal arithmetic prime $P=(\alpha)\in\mathcal{A}_{\mathbb{T}}$, we have the following canonical exact sequence up to finite error of Hecke modules:*

$$0 \to \widehat{J}^{\mathrm{ord}}_\infty(k)_{\mathbb{T}} \xrightarrow{\alpha} \widehat{J}^{\mathrm{ord}}_\infty(k)_{\mathbb{T}} \xrightarrow{\rho_\infty} \widehat{B}^{\mathrm{ord}}_P(k)_{\mathbb{T}}. \tag{1.1}$$

This theorem will be proven as Theorem 9.2. The exact sequence in the theorem is a Mordell–Weil analogue of a result of Nekovář in [N06, 12.7.13.4] for Selmer groups and implies that $\widehat{J}^{\mathrm{ord}}_\infty(k)$ is a Λ-module of finite type. In the text, we prove a stronger result showing finiteness of $\mathrm{Coker}(\rho_\infty)$ for almost all principal arithmetic primes P if the ordinary part of Selmer group of B_{P_0} is finite for one principal arithmetic prime P_0 (see Theorem 10.1).

Put $\check{J}_\infty(k)^*_{\mathrm{ord}} := \mathrm{Hom}_{\mathbb{Z}_p}(\check{J}_\infty(k)^{\mathrm{ord}},\mathbb{Z}_p)$. In [H15, Theorem 1.1], we proved the following exact sequence:

$$\check{J}_\infty(k)^*_{\mathrm{ord},P} \xrightarrow{\alpha} \check{J}_\infty(k)^*_{\mathrm{ord},P} \to \widehat{A}_P(k)^*_{\mathrm{ord},P} \to 0$$

for arithmetic P of weight 2, in addition to the finiteness of $\check{J}_\infty(k)^*_{\mathrm{ord}}$ as a Λ-module. This sequence is a localization at P of the natural one. The two sequences could be dual each other if we have a Λ-adic version of the Néron–Tate height pairing.

Here is some notation for Hecke algebras used throughout the paper. Let

$$h_r(\mathbb{Z}) = \mathbb{Z}[T(n), U(l) : l|Np, (n, Np) = 1] \subset \mathrm{End}(J_r),$$

and put $h_r(R) = h_r(\mathbb{Z}) \otimes_{\mathbb{Z}} R$ for any commutative ring R. Then we define $\mathbf{h}_r = e(h_r(\mathbb{Z}_p))$. The restriction morphism $h_s(\mathbb{Z}) \ni h \mapsto h|_{J_r} \in h_r(\mathbb{Z})$ for $s > r$ induces a projective system $\{\mathbf{h}_r\}_r$ whose limit gives rise to a big ordinary Hecke algebra

$$\mathbf{h} = \mathbf{h}(N) := \varprojlim_r \mathbf{h}_r.$$

Writing $\langle l\rangle$ (the diamond operator) for the action of $l \in (\mathbb{Z}/Np^r\mathbb{Z})^\times = \mathrm{Gal}(X_r/X_0(Np^r))$, we have an identity $l\langle l\rangle = T(l)^2 - T(l^2) \in h_r(\mathbb{Z}_p)$ for all primes $l \nmid Np$. Thus we have a canonical Λ-algebra structure $\Lambda = \mathbb{Z}_p[[\Gamma]] \hookrightarrow \mathbf{h}$. It is now well known that $\mathbf{h}$ is a free of finite rank over Λ and $\mathbf{h}_r = \mathbf{h} \otimes_\Lambda \Lambda/(\gamma^{p^{r-\epsilon}} - 1)$ (cf. [H86a]). Though the construction of the big Hecke algebra is intrinsic, to relate an algebra homomorphism $\lambda : \mathbf{h} \to \overline{\mathbb{Q}}_p$ killing $\gamma^{p^r} - 1$ for $r > 0$ to a classical Hecke eigenform, we need to fix (once and for all) an embedding $\overline{\mathbb{Q}} \xrightarrow{i_p} \overline{\mathbb{Q}}_p$ of the algebraic closure $\overline{\mathbb{Q}}$ in $\mathbb{C}$ into a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We write i_∞ for the inclusion $\overline{\mathbb{Q}} \subset \mathbb{C}$.

The following two sections Sections 3 and 4 (after a description of sheaves associated to abelian varieties) about $U(p)$-isomorphisms are an expanded version of a conference talk at CRM (see `http://www.crm.umontreal.ca/Representations05/indexen.html`) in September of 2005 which was not posted in the author's web page, though the lecture notes of the two lectures [H05] at CRM earlier than the conference have been posted. While converting [H05] into a research article [H14], the author found an application to Mordell–Weil groups of modular Jacobians. The author is grateful for CRM's invitation to speak. The author would like to thank the referee of this paper for careful reading (and the proof of (10.4) in the old version is incomplete as was pointed out by the referee). Heuristically, as explained just after Theorem 10.1, this point does not cause much trouble as we are dealing with the standard tower for which the root number for members of the family is not equal to -1 for most arithmetic point; so, presumably, the Mordell Weil group of B_P is finite for most P.

Contents

2. Sheaves associated to abelian varieties

Here is a general fact proven in [H15, §2] about sheaves associated to abelian varieties. Let $0 \to A \to B \to C \to 0$ be an exact sequence of algebraic groups proper over a field k. The field k is either a number field or a finite extension of the l-adic field $\mathbb{Q}_l$ for a prime l. We assume that B and C are abelian varieties. However A can be an extension of an abelian variety by a finite (étale) group.

If k is a number field, let S be a set of places including all archimedean places of k such that all members of the above exact sequence have good reduction outside S. We use the symbol K for k^S (the maximal extension unramified outside S) if k is a number field and for $\overline{k}$ (an algebraic closure of k) if k is a finite extension of $\mathbb{Q}_l$. A general field extension of k is denoted by κ. We consider the étale topology, the smooth topology and the fppf topology on the small site over $\mathrm{Spec}(k)$. Here under the smooth topology, covering families are made of faithfully flat smooth morphisms.

For the moment, assume that k is a number field. In this case, for an extension X of abelian variety defined over k by a finite étale group scheme, we define $\widehat{X}(\kappa) := X(\kappa) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for an fppf extension κ over k. By Mordell–Weil theorem (and its extension to fields of finite type over $\mathbb{Q}$; e.g., [RTP, IV]), we have $\widehat{X}(\kappa) = \varprojlim_n X(\kappa)/p^n X(\kappa)$ if κ is a field extension of k of finite type. We may regard the sequence $0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$ as an exact sequence of fppf abelian sheaves over k (or over any subring of k over which B and C extends to abelian schemes). Since we find a complementary abelian subvariety C' of B such that C' is isogenous to C and $B = A + C'$ with finite $A \cap C'$, adding the primes dividing the order $|A \cap C'|$ to S, the intersection $A \cap C' \cong \mathrm{Ker}(C' \to C)$ extends to an étale finite group scheme outside S; so, $C'(K) \to C(K)$ is surjective. Thus we have an exact sequence of $\mathrm{Gal}(K/k)$-modules

$$0 \to A(K) \to B(K) \to C(K) \to 0.$$

Note that $\widehat{A}(K) = A(K) \otimes_{\mathbb{Z}} \mathbb{Z}_p := \bigcup_F \widehat{A}(F)$ for F running over all finite extensions of k inside K. Then we have an exact sequence

$$0 \to \widehat{A}(K) \to \widehat{B}(K) \to \widehat{C}(K) \to 0. \tag{2.1}$$

Now assume that k is a finite extension of $\mathbb{Q}_l$. Again we use F to denote a finite field extension of k. Then $A(F) \cong O_F^{\dim A} \oplus \Delta_F$ for a finite group Δ_F for the l-adic integer ring O_F of F (by [M55] or [T66]). Thus if $l \neq p$, $\widehat{A}(F) := \varprojlim_n A(F)/p^n A(F) = \Delta_F \otimes_{\mathbb{Z}} \mathbb{Z}_p = A[p^\infty](F)$. Recall $K = \overline{k}$. Then $\widehat{A}(K) = A[p^\infty](K)$ (for $A[p^\infty] = \varinjlim_n A[p^n]$ with $A[p^n] = \mathrm{Ker}(p^n : A \to A)$); so, defining $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ by $A[p^\infty]$, $B[p^\infty]$ and $C[p^\infty]$ as fppf abelian sheaves, we again have the exact sequence (2.1) of $\mathrm{Gal}(\overline{k}/k)$-modules:

$$0 \to \widehat{A}(K) \to \widehat{B}(K) \to \widehat{C}(K) \to 0$$

and an exact sequence of fppf abelian sheaves

$$0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$$

whose value at a finite field extension $\kappa/\mathbb{Q}_l$ coincides with $\widehat{X}(\kappa) = \varprojlim_n X(\kappa)/p^n X(\kappa)$ for $X = A, B, C$.

Suppose $l = p$. For any module M, we define $M^{(p)}$ by the maximal prime-to-p torsion submodule of M. For $X = A, B, C$ and an fppf extension $R_{/k}$, the sheaf $R \mapsto X^{(p)}(R) = \varinjlim_{p\nmid N} X[N](R)$ is an fppf abelian sheaf. Then we define the fppf abelian sheaf $\widehat{X}$ by the sheaf quotient $X/X^{(p)}$. Since $X(F) = O_F^{\dim X} \oplus X[p^\infty](F) \oplus X^{(p)}(F)$ for a finite field extension $F_{/k}$, over the étale site on k, $\widehat{X}$ is the sheaf associated to a presheaf $R \mapsto O_F^{\dim X} \oplus X[p^\infty](R)$. If X has semi-stable reduction over O_F, we have $\widehat{X}(F) = X^\circ(O_F) + X[p^\infty](F) \subset X(F)$ for the formal group X° of the identity connected component of the Néron model of X over O_F [T66]. Since X becomes semi-stable over a finite Galois extension F_0/k, in general $\widehat{X}(F) = H^0(\mathrm{Gal}(F_0F/F), X(F_0F))$ for any finite extension $F_{/K}$ (or more generally for each finite étale extension $F_{/k}$); so, $F \mapsto \widehat{X}(F)$ is a sheaf over the étale site on k. Thus by [ECH, II.1.5], the sheafication coincides over the étale site with the presheaf $F \mapsto \varprojlim_n X(F)/p^nX(F)$. Thus we conclude $\widehat{X}(F) = \varprojlim_n X(F)/p^nX(F)$ for any étale finite extensions $F_{/k}$. Moreover $\widehat{X}(K) = \bigcup_{K/F/k} \widehat{X}(F)$. Applying the snake lemma to the commutative diagram with exact rows (in the category of fppf abelian sheaves):

$$\begin{array}{ccccc} A^{(p)} & \xrightarrow{\hookrightarrow} & B^{(p)} & \xrightarrow{\twoheadrightarrow} & C^{(p)} \\ \cap\downarrow & & \cap\downarrow & & \cap\downarrow \\ A & \underset{\hookrightarrow}{\longrightarrow} & B & \underset{\twoheadrightarrow}{\longrightarrow} & C, \end{array}$$

the cokernel sequence gives rise to an exact sequence of fppf abelian sheaves over k:

$$0 \to \widehat{A} \to \widehat{B} \to \widehat{C} \to 0$$

and an exact sequence of $\mathrm{Gal}(\overline{k}/k)$-modules

$$0 \to \widehat{A}(K) \to \widehat{B}(K) \to \widehat{C}(K) \to 0.$$

In this way, we extended the sheaves $\widehat{A}, \widehat{B}, \widehat{C}$ to fppf abelian sheaves keeping the exact sequence $\widehat{A} \hookrightarrow \widehat{B} \twoheadrightarrow \widehat{C}$ intact. However note that our way of defining $\widehat{X}$ for $X = A, B, C$ depends on the base field $k = \mathbb{Q}, \mathbb{Q}_p, \mathbb{Q}_l$. Here is a summary for fppf algebras $R_{/k}$:

$$\text{(S)} \qquad \widehat{X}(R) = \begin{cases} X(R) \otimes_{\mathbb{Z}} \mathbb{Z}_p & \text{if } [k:\mathbb{Q}] < \infty, \\ X[p^\infty](R) & \text{if } [k:\mathbb{Q}_l] < \infty \ (l \neq p), \\ (X/X^{(p)})(R) \text{ as a sheaf quotient} & \text{if } [k:\mathbb{Q}_p] < \infty. \end{cases}$$

Here is a sufficient condition when $\widehat{X}(\kappa)$ is given by the projective limit: $\varprojlim_n X(\kappa)/p^nX(\kappa)$ for $X = A, B$ or C:

(2.2)

$$\widehat{X}(\kappa) = \varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa) \ \text{ if } \begin{cases} [k:\mathbb{Q}] < \infty \text{ and } \kappa \text{ is a field of finite type over } k \\ [k:\mathbb{Q}_l] < \infty \text{ with } l \neq p \text{ and } \kappa \text{ is a field of finite type over } k \\ [k:\mathbb{Q}_p] < \infty \text{ and } \kappa \text{ is a finite algebraic extension over } k. \end{cases}$$

A slightly weaker sufficient condition for $\widehat{X}(\kappa) = \varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa)$ is proven in [H15, Lemma 2.1].

For a sheaf X under the topology ?, we write $H^\bullet_?(X)$ for the cohomology group $H^1_?(\mathrm{Spec}(k), X)$ under the topology ?. If we have no subscript, $H^1(X)$ means the Galois cohomology $H^\bullet(\mathrm{Gal}(K/k), X)$ for the $\mathrm{Gal}(K/k)$-module X. For any $\mathbb{Z}_p$-module M, we put $T_pM = \varprojlim_n M[p^n] = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, M)$.

The following fact is essentially proven in [H15, Lemma 2.2] (where it was proven for finite S but the same proof works for infinite S as is obvious from the fact that it works under fppf topology):

Lemma 2.1. *Let X be an extension of an abelian variety over k by a finite étale group scheme of order prime to p. Then, we have a canonical injection*

$$\varprojlim_n \widehat{X}(k)/p^n\widehat{X}(k) \hookrightarrow \varprojlim_n H^1(X[p^n]).$$

Similarly, for any fppf or smooth extension κ/k of finite type which is an integral domain, we have an injection

$$\varprojlim_n \widehat{X}(\kappa)/p^n\widehat{X}(\kappa) \hookrightarrow \varprojlim_n H^1_?(\mathrm{Spec}(\kappa), X[p^n])$$

for $? = \mathrm{fppf}$ *or* sm *according as κ/k is an fppf extension or a smooth extension of finite type. For Galois cohomology, we have an exact sequence for $j = 0, 1$:*

$$0 \to \varprojlim_n H^j(\widehat{X}(k))/p^n H^j(\widehat{X}(k)) \to \varprojlim_n H^{j+1}(X[p^n]) \to T_pH^{j+1}(X).$$

The natural map: $\varprojlim_n H^{j+1}(X[p^n]) \xrightarrow{\pi} T_pH^{j+1}(X)$ is surjective if either $j = 0$ or k is local or S is finite. In particular, $H^1(T_pX)$ for $T_pX = \varprojlim_n X[p^n]$ is equal to $\varprojlim_n H^1(X[p^n])$, and

$$0 \to \widehat{X}(k) \to H^1(T_pX) \to T_pH^1(X)[p^n] \to 0$$

is exact.

We shall give a detailed proof of the surjectivity of π for Galois cohomology (which we will use) along with a sketch of the proof of the exactness.

Proof. By p-divisibility, we have the sheaf exact sequence under the étale topology over $\mathrm{Spec}(\kappa)$

$$0 \to X[p^n] \to X \xrightarrow{p^n} X \to 0.$$

This implies, we have an exact sequence

$$0 \to X[p^n](K) \to X(K) \xrightarrow{p^n} X(K) \to 0.$$

By the long exact sequence associated to this sequence, for a finite intermediate extension $K/\kappa/k$, we have exactness of

$$(*) \qquad 0 \to H^j(X(\kappa))/p^n H^j(X(\kappa)) \to H^{j+1}(X[p^n]) \to H^{j+1}(X)[p^n] \to 0.$$

Passing to the limit (with respect to n), we have the exactness of

$$0 \to \varprojlim_n H^j(X(\kappa))/p^n H^j(X(\kappa)) \to H^{j+1}(T_pX) \to T_pH^{j+1}(X).$$

as $\varprojlim_n H^{j+1}(X[p^n]) = H^{j+1}(\varprojlim_n X[p^n]) = H^j(T_pX)$ for $j = 0, 1$ without assumption if $j = 0$ and assuming S is finite if $j = 1$ (because of finiteness of $X[p^n](K)$ and p-divisibility of X; e.g., [CNF, Corollary 2.7.6] and [H16, Lemma 7.1 (2)]).

Assume $\kappa = k$. If k is local or S is finite, by Tate duality, all the terms of $(*)$ is finite; so, the surjectivity of $(*)$ is kept after passing to the limit. If $j = 0$ and $\kappa = k$, $X(k)/p^nX(k)$ is a finite module; so, the sequences $(*)$ satisfied Mittag–Leffler condition. Thus again the surjectivity of $(*)$ is kept after passing to the limit. □

For finite S, the following module structure of $H^1(\widehat{A})$ is well known (see [ADT, Corollary I.4.15] or [H15, Lemma 2.3]):

Lemma 2.2. *Let k be a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_l$ for a prime l. Suppose that S is finite if k is a finite extension of $\mathbb{Q}$. Let $A_{/k}$ be an abelian variety. Then $H^1(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p = H^1(\widehat{A})$ is isomorphic to the discrete module $(\mathbb{Q}_p/\mathbb{Z}_p)^r \oplus \Delta$ for a finite $r \geq 0$ and a finite p-torsion group Δ.*

Hereafter we assume that S is a finite set unless otherwise indicated.

3. $U(p)$-ISOMORPHISMS FOR GROUP COHOMOLOGY

For $\mathbb{Z}[U]$-modules X and Y, we call a $\mathbb{Z}[U]$-linear map $f : X \to Y$ a U-injection (resp. a U-surjection) if $\mathrm{Ker}(f)$ is killed by a power of U (resp. $\mathrm{Coker}(f)$ is killed by a power of U). If f is an U-injection and U-surjection, we call f is a U-isomorphism. If $X \to Y$ is a U-isomorphism, we write $X \cong_U Y$. In terms of U-isomorphisms (for $U = U(p), U^*(p)$, we describe briefly the facts we need in this article (and in later sections, we fill in more details in terms of the ordinary projector e and the co-ordinary projector $e^* := \lim_{n\to\infty} U^*(p)^{n!}$).

Let N be a positive integer prime to p. We consider the (open) modular curve $Y_1(Np^r)_{/\mathbb{Q}}$ which classifies elliptic curves E with an embedding $\phi : \mu_{p^r} \hookrightarrow E[p^r] = \mathrm{Ker}(p^r : E \to E)$ of finite flat groups. Let $R_i = \mathbb{Z}_{(p)}[\mu_{p^i}]$ and $K_i = \mathbb{Q}[\mu_{p^i}]$. For a valuation subring or a subfield R of K_∞ over $\mathbb{Z}_{(p)}$ with quotient field K, we write $X_{r/R}$ for the normalization of the j-line $\mathbf{P}(j)_{/R}$ in the function field of $Y_1(Np^r)_{/K}$. The group $z \in (\mathbb{Z}/p^r\mathbb{Z})^\times$ acts on X_r by $\phi \mapsto \phi \circ z$, as $\mathrm{Aut}(\mu_{Np^r}) \cong (\mathbb{Z}/Np^r\mathbb{Z})^\times$. Thus $\Gamma = 1 + p^\epsilon\mathbb{Z}_p = \gamma^{\mathbb{Z}_p}$ acts on X_r (and its Jacobian) through its image in $(\mathbb{Z}/Np^r\mathbb{Z})^\times$. Hereafter we take $U = U(p), U^*(p)$ for the Hecke–Atkin operator $U(p)$.

Let $J_{r/R} = \mathrm{Pic}^0_{X_{r/R}}$ be the connected component of the Picard scheme. We state a result comparing $J_{r/R}$ and the Néron model of $J_{r/K}$ over R. Thus we assume that R is a valuation ring. By [AME, 5.5.1, 13.5.6, 13.11.4], $X_{r/R}$ is regular; the reduction $X_r \otimes_R \mathbb{F}_p$ is a union of irreducible components, and the component containing the ∞ cusp has geometric multiplicity 1. Then by [NMD, Theorem 9.5.4], $J_{r/R}$ gives the identity connected component of the Néron model of the Jacobian of $X_{r/R}$. In this paper, we do not use these fine integral structure of $X_{r/R}$ but work with $X_{r/\mathbb{Q}}$. We just wanted to note these facts for possible use in our future articles.

We write $X^s_{r/R}$ for the normalization of the j-line of the canonical $\mathbb{Q}$-curve associated to the modular curve for the congruence subgroup $\Gamma^r_s = \Gamma_1(Np^r) \cap \Gamma_0(p^s)$ for $0 < r \le s$. We denote $\mathrm{Pic}^0_{X^r_{s/R}}$ by $J^r_{s/R}$. Similarly, as above, $J^r_{s/R}$ is the connected component of the Néron model of $X^r_{s/K}$. Note that, for $\alpha_m = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^m \end{smallmatrix}\right)$,

$$\Gamma^r_s\backslash\Gamma^r_s\alpha_{s-r}\Gamma_1(Np^r) = \left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & p^{s-r} \end{smallmatrix}\right) \middle| a \mod p^{s-r} \right\} = \Gamma_1(Np^r)\backslash\Gamma_1(Np^r)\alpha_{s-r}\Gamma_1(Np^r). \tag{3.1}$$

Write $U^s_r(p^{s-r}) : J^s_{r/R} \to J_{r/R}$ for the Hecke operator of $\Gamma^s_r\alpha_{s-r}\Gamma_1(Np^r)$. Strictly speaking, the Hecke operator induces a morphism of the generic fiber of the Jacobians and then extends to their connected components of the Néron models by the functoriality of the model (or by Picard functoriality). Then we have the following commutative diagram from the above identity, first over $\mathbb{C}$, then over K and by Picard functoriality over R:

$$\begin{array}{ccc} J_{r/R} & \xrightarrow{\pi^*} & J^r_{s/R} \\ \downarrow u & \swarrow u' & \downarrow u'' \\ J_{r/R} & \xrightarrow{\pi^*} & J^r_{s/R}, \end{array} \tag{3.2}$$

where the middle u' is given by $U^s_r(p^{s-r})$ and u and u'' are $U(p^{s-r})$. Thus

(u1) $\pi^* : J_{r/R} \to J^r_{s/R}$ is a $U(p)$-isomorphism (for the projection $\pi : X^r_s \to X_r$).

Taking the dual $U^*(p)$ of $U(p)$ with respect to the Rosati involution associated to the canonical polarization of the Jacobians, we have a dual version of the above diagram for $s > r > 0$:

$$\begin{array}{ccc} J_{r/R} & \xleftarrow{\pi_*} & J^r_{s/R} \\ \uparrow u^* & \nearrow u'^* & \uparrow u''^* \\ J_{r/R} & \xleftarrow{\pi_*} & J^r_{s/R}. \end{array} \tag{3.3}$$

Here the superscript "$*$" indicates the Rosati involution of the canonical divisor of the Jacobians, and $u^* = U^*(p)^{s-r}$ for the level $\Gamma_1(Np^r)$ and $u''^* = U^*(p)^{s-r}$ for Γ^r_s. Note that these morphisms come from the following coset decomposition, for $\beta_m := \left(\begin{smallmatrix} p^m & 0 \\ 0 & 1 \end{smallmatrix}\right)\Gamma_1(Np^r)$,

$$\Gamma^r_s\backslash\Gamma^r_s\beta_{s-r}\Gamma_1(Np^r) = \left\{ \left(\begin{smallmatrix} p^{s-r} & a \\ 0 & 1 \end{smallmatrix}\right) \middle| a \mod p^{s-r} \right\} = \Gamma_1(Np^r)\backslash\Gamma_1(Np^r)\beta_{s-r}\Gamma_1(Np^r). \tag{3.4}$$

From this, we get

(u*1) $\pi_* : J_{r/R} \to J^r_{s/R}$ is a $U^*(p)$-isomorphism, where π_* is the dual of π^*.

In particular, if we take the ordinary and the co-ordinary projector $e = \lim_{n\to\infty} U(p)^{n!}$ and $e^* = \lim_{n\to\infty} U^*(p)^{n!}$ on $J[p^\infty]$ for $J = J_{r/R}, J_{s/R}, J^r_{s/R}$, noting $U(p^m) = U(p)^m$, we have

$$\pi^* : J^{\mathrm{ord}}_{r/R}[p^\infty] \cong J^{r,\mathrm{ord}}_{s/R}[p^\infty] \text{ and } \pi_* : J^{r,\text{co-ord}}_{s/R}[p^\infty] \cong J^{\text{co-ord}}_{r/R}[p^\infty]$$

where "ord" (resp. "co-ord") indicates the image of the projector e (resp. e^*). For simplicity, we write $\mathcal{G}_{r/R} := J^{\mathrm{ord}}_{r/R}[p^\infty]_{/R}$, and we set $\mathcal{G} := \varinjlim_r \mathcal{G}_r$.

Pick a congruence subgroup Γ defining the modular curve $X(\mathbb{C}) = \Gamma\backslash(\mathfrak{H} \sqcup \mathbf{P}^1(\mathbb{Q}))$, and write its Jacobian as J. We now identify $J(\mathbb{C})$ with a subgroup of $H^1(\Gamma, \mathbf{T})$ (for the trivial Γ-module $\mathbf{T} := \mathbb{R}/\mathbb{Z} \cong \{z \in \mathbb{C}^\times : |z| = 1\}$ with trivial Γ-action). Since $\Gamma^r_s \rhd \Gamma_1(Np^s)$, consider the finite cyclic quotient group $C := \frac{\Gamma^r_s}{\Gamma_1(Np^s)}$. By the inflation restriction sequence, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^1(C,\mathbf{T}) & \xrightarrow{\hookrightarrow} & H^1(\Gamma^r_s,\mathbf{T}) & \longrightarrow & H^1(\Gamma_1(Np^s),\mathbf{T})^{\gamma^{p^r}=1} & \longrightarrow & H^2(C,\mathbf{T}) \\ \uparrow & & \cup\uparrow & & \uparrow\cup & & \uparrow \\ ? & \longrightarrow & J^r_s(\mathbb{C}) & \longrightarrow & J_s(\mathbb{C})[\gamma^{p^{r-\epsilon}}-1] & \longrightarrow & ?. \end{array} \tag{3.5}$$

Since C is a finite cyclic group of order p^{s-r} (with generator g) acting trivially on $\mathbf{T}$, we have $H^1(C,\mathbf{T}) = \mathrm{Hom}(C,\mathbf{T}) \cong C$ and

$$H^2(C,\mathbf{T}) = \mathbf{T}/(1+g+\cdots+g^{p^{s-r}-1}) = \mathbf{T}/p^{s-r}\mathbf{T} = 0.$$

By the same token, replacing $\mathbf{T}$ by $\mathbb{T}_p := \mathbb{Q}_p/\mathbb{Z}_p$, we get $H^2(C,\mathbb{T}_p) = 0$. By a sheer computation (cf. [H86b, Lemma 6.1]), we confirm that $U(p)$ acts on $H^1(C,\mathbf{T})$ and $H^1(C,\mathbb{T}_p)$ via multiplication by its degree p, and hence $U(p)^{s-r}$ kill $H^1(C,\mathbf{T})$ and $H^1(C,\mathbb{T}_p)$. We record what we have proven:

$$U(p)^{s-r}(H^1(C,\mathbb{T}_p)) = H^2(C,\mathbf{T}) = H^2(C,\mathbb{T}_p) = 0. \tag{3.6}$$

This fact has been exploited by the author (for example, [H86b] and [H14]) to study the modular Barsotti–Tate groups $J_s[p^\infty]$.

4. $U(p)$-isomorphisms for arithmetic cohomology

To good extent, we reproduce the results and proofs in [H15, §3] as it is important in the sequel. Let $X \to Y \to S$ be proper morphisms of noetherian schemes. We now replace $H^1(\Gamma,\mathbf{T})$ in the above diagram (3.5) by

$$H^0_{\mathrm{fppf}}(T, R^1 f_*\mathbb{G}_m) = R^1 f_* O^\times_X(T) = \mathrm{Pic}_{X/S}(T)$$

for S-scheme T and the structure morphism $f : X \to S$, and do the same analysis as in Section 3 for arithmetic cohomology in place of group cohomology (via the moduli theory of Katz-Mazur and Drinfeld; cf., [AME]). Write the morphisms as $X \xrightarrow{\pi} Y \xrightarrow{g} S$ with $f = g \circ \pi$. Assume that π is finite flat.

Suppose that f and g have compatible sections $S \xrightarrow{s_g} Y$ and $S \xrightarrow{s_f} X$ so that $\pi \circ s_f = s_g$. Then we get (e.g., [NMD, Section 8.1])

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Ker}(s^1_f : H^1_{\mathrm{fppf}}(X_T, O^\times_X) \to H^1_{\mathrm{fppf}}(T, O^\times_T))$$
$$\mathrm{Pic}_{Y/S}(T) = \mathrm{Ker}(s^1_g : H^1_{\mathrm{fppf}}(Y_T, O^\times_{Y_T}) \to H^1_{\mathrm{fppf}}(T, O^\times_T))$$

for any S-scheme T, where $s^q_f : H^q(X_T, O^\times_{X_T}) \to H^q(T, O^\times_T)$ and $s^n_g : H^n(Y_T, O^\times_{Y_T}) \to H^n(T, O^\times_T)$ are morphisms induced by s_f and s_g, respectively. Here $X_T = X \times_S T$ and $Y_T = Y \times_S T$. We suppose that the functors $\mathrm{Pic}_{X/S}$ and $\mathrm{Pic}_{Y/S}$ are representable by group schemes whose connected components are smooth (for example, if X, Y are curves and $S = \mathrm{Spec}(k)$ for a field k; see [NMD, Theorem 8.2.3 and Proposition 8.4.2]). We then put $J_? = \mathrm{Pic}^0_{?/S}$ $(? = X, Y)$. Anyway we suppose hereafter also that X, Y, S are varieties (in the sense of [ALG, II.4]).

For an fppf covering $\mathcal{U} \to Y$ and a presheaf $P = P_Y$ on the fppf site over Y, we define via Čech cohomology theory an fppf presheaf $\mathcal{U} \mapsto \check{H}^q(\mathcal{U}, P)$ denoted by $\underline{\check{H}}^q(P_Y)$ (see [ECH, III.2.2 (b)]). The inclusion functor from the category of fppf sheaves over Y into the category of fppf presheaves over Y is left exact. The derived functor of this inclusion of an fppf sheaf $F = F_Y$ is denoted by $\underline{H}^\bullet(F_Y)$ (see [ECH, III.1.5 (c)]). Thus $\underline{H}^\bullet(\mathbb{G}_{m/Y})(\mathcal{U}) = H^\bullet_{\mathrm{fppf}}(\mathcal{U}, O^\times_{\mathcal{U}})$ for a Y-scheme $\mathcal{U}$ as a presheaf (here $\mathcal{U}$ varies in the small fppf site over Y).

Instead of the Hochschild-Serre spectral sequence producing the top row of the diagram (3.5), assuming that f, g and π are all faithfully flat of finite presentation, we use the spectral sequence of Čech cohomology of the flat covering $\pi : X \twoheadrightarrow Y$ in the fppf site over Y [ECH, III.2.7]:

$$\check{H}^p(X_T/Y_T, \underline{H}^q(\mathbb{G}_{m/Y})) \Rightarrow H^n_{\mathrm{fppf}}(Y_T, O^\times_{Y_T}) \underset{\iota}{\xrightarrow{\sim}} H^n(Y_T, O^\times_{Y_T}) \tag{4.1}$$

for each S-scheme T. Here $F \mapsto H^n_{\mathrm{fppf}}(Y_T, F)$ (resp. $F \mapsto H^n(Y_T, F)$) is the right derived functor of the global section functor: $F \mapsto F(Y_T)$ from the category of fppf sheaves (resp. Zariski sheaves) over Y_T to the category of abelian groups. The canonical isomorphism ι is the one given in [ECH, III.4.9].

By the sections $s_?$, we have a splitting $H^q(X_T, O^\times_{X_T}) = \mathrm{Ker}(s^q_f) \oplus H^q(T, O^\times_T)$ and $H^n(Y_T, O^\times_{Y_T}) = \mathrm{Ker}(s^n_g) \oplus H^n(T, O^\times_T)$. Write $\underline{H}^\bullet_{Y_T}$ for $\underline{H}^\bullet(\mathbb{G}_{m/Y_T})$ and $\check{H}^\bullet(\underline{H}^0_{Y_T})$ for $\check{H}^\bullet(Y_T/X_T, \underline{H}^0_{Y_T})$. Since

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Ker}(s^1_{f,T} : H^1(X_T, O^\times_{X_T}) \to H^1(T, O^\times_T))$$

for the morphism $f : X \to S$ with a section [NMD, Proposition 8.1.4], from the spectral sequence (4.1), we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\check{H}^1(\underline{H}^0_{Y_T}) & \xrightarrow{\hookrightarrow} & H^1(T, O^\times_T) \oplus \mathrm{Ker}(s^1_{g,T}) & \xrightarrow{a} & \check{H}^0(\frac{X_T}{Y_T}, \underline{H}^1(\mathbb{G}_{m,Y})) & \longrightarrow & \check{H}^2(\underline{H}^0_{Y_T}) \\
\| \uparrow & & \wr \uparrow & & \uparrow \| & & \uparrow \| \\
\check{H}^1(\underline{H}^0_{Y_T}) & \longrightarrow & \mathrm{Pic}_T \oplus \mathrm{Pic}_{Y/S}(T) & \xrightarrow{b} & \check{H}^0(\frac{X_T}{Y_T}, \mathrm{Pic}_Y(T)) & \longrightarrow & \check{H}^2(\underline{H}^0_{Y_T}) \\
\uparrow & & \cup \uparrow & & \uparrow \cup & & \uparrow \\
?_1 & \longrightarrow & \mathrm{Pic}_T \oplus J_Y(T) & \underset{c}{\longrightarrow} & \mathrm{Pic}_T \oplus \check{H}^0(\frac{X_T}{Y_T}, J_X(T)) & \longrightarrow & ?_2,
\end{array} \tag{4.2}$$

where we have written $J_? = \mathrm{Pic}^0_{?/S}$ (the identity connected component of $\mathrm{Pic}_{?/S}$). Here the horizontal exactness at the top two rows follows from the spectral sequence (4.1) (see [ECH, Appendix B]).

Take a correspondence $U \subset Y \times_S Y$ given by two finite flat projections $\pi_1, \pi_2 : U \to Y$ of constant degree (i.e., $\pi_{j,*}\mathcal{O}_U$ is locally free of finite rank $\deg(\pi_j)$ over $\mathcal{O}_Y$). Consider the pullback $U_X \subset X \times_S X$ given by the Cartesian diagram:

$$\begin{array}{ccc}
U_X = U \times_{Y \times_S Y} (X \times_S X) & \longrightarrow & X \times_S X \\
\downarrow & & \downarrow \\
U & \xrightarrow{\hookrightarrow} & Y \times_S Y
\end{array}$$

Let $\pi_{j,X} = \pi_j \times_S \pi : U_X \twoheadrightarrow X$ ($j = 1, 2$) be the projections.

Consider a new correspondence $U^{(q)}_X = \overbrace{U_X \times_Y U_X \times_Y \cdots \times_Y U_X}^{q}$, whose projections are the iterated product

$$\pi_{j,X^{(q)}} = \pi_{j,X} \times_Y \cdots \times_Y \pi_{j,X} : U^{(q)}_X \to X^{(q)} \ (j = 1, 2).$$

Here is the first step to prove a result analogous to (3.6) for arithmetic cohomology.

Lemma 4.1. *Let the notation and the assumption be as above. In particular, $\pi : X \to Y$ is a finite flat morphism of geometrically reduced proper schemes over $S = \mathrm{Spec}(k)$ for a field k. Suppose that X and U_X are proper schemes over a field k satisfying one of the following conditions:*

1. *U_X is geometrically reduced, and for each geometrically connected component X° of X, its pull back to U_X by $\pi_{2,X}$ is also connected; i.e., $\pi^0(X) \underset{\sim}{\xrightarrow{\pi^*_{2,X}}} \pi^0(U_X)$;*
2. *$(f \circ \pi_{2,X})_*\mathcal{O}_{U_X} = f_*\mathcal{O}_X$.*

If $\pi_2 : U \to Y$ *has constant degree* $\deg(\pi_2)$*, then, for each* $q > 0$*, the action of* $U_X^{(q)}$ *on* $H^0(X^{(q)}, \mathcal{O}_{X^{(q)}}^\times)$ *factors through the multiplication by* $\deg(\pi_2) = \deg(\pi_{2,X})$*.*

This result is given as [H15, Lemma 3.1, Corollary 3.2].

To describe the correspondence action of U on $H^0(X, \mathcal{O}_X^\times)$ in down-to-earth terms, let us first recall the Čech cohomology: for a general S-scheme T,

$$(4.3)\quad \check{H}^q(\frac{X_T}{Y_T}, \underline{H}^0(\mathbb{G}_{m/Y})) = \frac{\{(c_{i_0,\dots,i_q}) | c_{i_0,\dots,i_q} \in H^0(X_T^{(q+1)}, O_{X_T^{(q+1)}}^\times) \text{ and } \prod_j (c_{i_0\dots\check{i}_j\dots i_{q+1}} \circ p_{i_0\dots\check{i}_j\dots i_{q+1}})^{(-1)^j} = 1\}}{\{db_{i_0\dots i_q} = \prod_j (b_{i_0\dots\check{i}_j\dots i_q} \circ p_{i_0\dots\check{i}_j\dots i_q})^{(-1)^j} | b_{i_0\dots\check{i}_j\dots i_q} \in H^0(X_T^{(q)}, O_{X_T^{(q)}}^\times)\}}$$

where we agree to put $H^0(X_T^{(0)}, O_{X_T}^{(0)}) = 0$ as a convention,

$$X_T^{(q)} = \overbrace{X \times_Y X \times_Y \cdots \times_Y X}^{q} \times_S T, O_{X_T^{(q)}} = \overbrace{O_X \times_{O_Y} O_X \times_{O_Y} \cdots \times_{O_Y} O_X}^{q} \times_{O_S} O_T,$$

the identity $\prod_j (c \circ p_{i_0\dots\check{i}_j\dots i_{q+1}})^{(-1)^j} = 1$ takes place in $O_{X_T^{(q+2)}}$ and $p_{i_0\dots\check{i}_j\dots i_{q+1}} : X_T^{(q+2)} \to X_T^{(q+1)}$ is the projection to the product of X the j-th factor removed. Since $T \times_T T \cong T$ canonically, we have $X_T^{(q)} \cong \overbrace{X_T \times_T \cdots \times_T X_T}^{q}$ by transitivity of fiber product.

Consider $\alpha \in H^0(X, \mathcal{O}_X)$. Then we lift $\pi_{1,X}^*\alpha = \alpha \circ \pi_{1,X} \in H^0(U_X, \mathcal{O}_{U_X})$. Put $\alpha_U := \pi_{1,X}^*\alpha$. Note that $\pi_{2,X,*}\mathcal{O}_{U_X}$ is locally free of rank $d = \deg(\pi_2)$ over $\mathcal{O}_X$, the multiplication by α_U has its characteristic polynomial $P(T)$ of degree d with coefficients in $\mathcal{O}_X$. We define the norm $N_U(\alpha_U)$ to be the constant term $P(0)$. Since α is a global section, $N_U(\alpha_U)$ is a global section, as it is defined everywhere locally. If $\alpha \in H^0(X, \mathcal{O}_X^\times)$, $N_U(\alpha_U) \in H^0(X, \mathcal{O}_X^\times)$. Then define $U(\alpha) = N_U(\alpha_U)$, and in this way, U acts on $H^0(X, \mathcal{O}_X^\times)$.

For a degree q Čech cohomology class $[c] \in \check{H}^q(X_{/Y}, \underline{H}^0(\mathbb{G}_{m/Y}))$ with a Čech q-cocycle $c = (c_{i_0,\dots,i_q})$, $U([c])$ is given by the cohomology class of the Čech cocycle $U(c) = (U(c_{i_0,\dots,i_q}))$, where $U(c_{i_0,\dots,i_q})$ is the image of the global section $c_{i_0,\dots,i_q}$ under U. Indeed, $(\pi_{1,X}^* c_{i_0,\dots,i_q})$ plainly satisfies the cocycle condition, and $(N_U(\pi_{1,X}^* c_{i_0,\dots,i_q}))$ is again a Čech cocycle as N_U is a multiplicative homomorphism. By the same token, this operation sends coboundaries to coboundaries, and define the action of U on the cohomology group. We get the following vanishing result (cf. (3.6)):

Proposition 4.2. *Suppose that* $S = \mathrm{Spec}(k)$ *for a field* k*. Let* $\pi : X \to Y$ *be a finite flat covering of (constant) degree* d *of geometrically reduced proper varieties over* k*, and let* $Y \xleftarrow{\pi_1} U \xrightarrow{\pi_2} Y$ *be two finite flat coverings (of constant degree) identifying the correspondence* U *with a closed subscheme* $U \overset{\pi_1\times\pi_2}{\hookrightarrow} Y \times_S Y$*. Write* $\pi_{j,X} : U_X = U \times_Y X \to X$ *for the base-change to* X*. Suppose one of the conditions* (1) *and* (2) *of* Lemma 4.1 *for* (X, U)*. Then*

(1) *The correspondence* $U \subset Y \times_S Y$ *sends* $\check{H}^q(\underline{H}_Y^0)$ *into* $\deg(\pi_2)(\check{H}^q(\underline{H}_Y^0))$ *for all* $q > 0$*.*
(2) *If* d *is a* p*-power and* $\deg(\pi_2)$ *is divisible by* p*,* $\check{H}^q(\underline{H}_Y^0)$ *for* $q > 0$ *is killed by* U^M *if* $p^M \geq d$*.*
(3) *The cohomology* $\check{H}^q(\underline{H}_Y^0)$ *with* $q > 0$ *is killed by* d*.*

This follows from Lemma 4.1, because on each Čech q-cocycle (whose value is a global section of iterated product $X_T^{(q+1)}$), the action of U is given by $U^{(q+1)}$ by (4.3). See [H15, Proposition 3.3] for a detailed proof. We can apply the above proposition to $(U, X, Y) = (U(p), X_s, X_s^r)$ with U given by $U(p) \subset X_s^r \times X_s^r$ over $\mathbb{Q}$. Indeed, $U := U(p) \subset X_s^r \times X_s^r$ corresponds to $X(\Gamma)$ given by $\Gamma = \Gamma_1(Np^r) \cap \Gamma_0(p^{s+1})$ and U_X is given by $X(\Gamma')$ for $\Gamma' = \Gamma_1(Np^s) \cap \Gamma_0(p^{s+1})$ both geometrically irreducible curves. In this case π_1 is induced by $z \mapsto \frac{z}{p}$ on the upper complex plane and π_2 is the natural projection of degree p. In this case, $\deg(X_s/X_s^r) = p^{s-r}$ and $\deg(\pi_2) = p$.

An easy criterion to see $\pi^0(U_X^{(q)}) = \pi^0(X^{(q)})$ (which will not be used in this paper), we can offer

Lemma 4.3. *For a finite flat covering $V \xrightarrow{\pi} X \xrightarrow{f} Y$ of geometrically irreducible varieties over a field k, if a fiber $f \circ \pi$ of a k-closed point $y \in Y$ of V is made of a single closed point $v \in V(k)$ (as a topological space), then $V^{(q)} := \overbrace{V \times_Y V \times_Y \cdots \times_Y V}^{q}$ and $X^{(q)}$ are geometrically connected.*

Proof. The q-fold tensor product of the stalks at v given by

$$\mathcal{O}_{V,v}^{(q)} := \overbrace{\mathcal{O}_{V,v} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{V,v} \otimes_{\mathcal{O}_{Y,y}} \cdots \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{V,v}}^{q}$$

is a local ring whose residue field is that of y. This fact holds true for the base change $V_{/k'} \to X_{/k'} \to Y_{/k'}$ for any algebraic extension k'/k; so, $V^{(q)}$ and $X^{(q)}$ are geometrically connected □

Assume that a finite group G acts on $X_{/Y}$ faithfully. Then we have a natural morphism $\phi : X \times G \to X \times_Y X$ given by $\phi(x,\sigma) = (x, \sigma(x))$. In other words, we have a commutative diagram

$$\begin{array}{ccc} X \times G & \xrightarrow{(x,\sigma)\mapsto\sigma(x)} & X \\ {\scriptstyle (x,\sigma)\mapsto x}\downarrow & & \downarrow \\ X & \longrightarrow & Y, \end{array}$$

which induces $\phi : X \times G \to X \times_Y X$ by the universality of the fiber product. Suppose that ϕ is surjective; for example, if Y is a geometric quotient of X by G; see [GME, §1.8.3]). Under this map, for any fppf abelian sheaf F, we have a natural map $\check{H}^0(X/Y, F) \to H^0(G, F(X))$ sending a Čech 0-cocycle $c \in H^0(X, F) = F(X)$ (with $p_1^* c = p_2^* c$) to $c \in H^0(G, F(X))$. Obviously, by the surjectivity of ϕ, the map $\check{H}^0(X/Y, F) \to H^0(G, F(X))$ is an isomorphism (e.g., [ECH, Example III.2.6, page 100]). Thus we get

Lemma 4.4. *Let the notation be as above, and suppose that ϕ is surjective. For any scheme T fppf over S, we have a canonical isomorphism: $\check{H}^0(X_T/Y_T, F) \cong H^0(G, F(X_T))$.*

We now assume $S = \mathrm{Spec}(k)$ for a field k and that X and Y are proper reduced connected curves. Then we have from the diagram (4.2) with the exact middle two columns and exact horizontal rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & = & \mathbb{Z} & \longrightarrow & 0 \\ \uparrow & & {\scriptstyle \deg}\uparrow{\scriptstyle \text{onto}} & & {\scriptstyle \deg}\uparrow{\scriptstyle \text{onto}} & & \uparrow \\ \check{H}^1(\underline{H}_Y^0) & \longrightarrow & \mathrm{Pic}_{Y/S}(T) & \xrightarrow{b} & \check{H}^0(\frac{X_T}{Y_T}, \mathrm{Pic}_{Y/S}(T)) & \longrightarrow & \check{H}^2(\underline{H}_Y^0) \\ \uparrow & & \cup\uparrow & & \uparrow\cup & & \uparrow \\ ?_1 & \longrightarrow & J_Y(T) & \xrightarrow[c]{} & \check{H}^0(\frac{X_T}{Y_T}, J_X(T)) & \longrightarrow & ?_2, \end{array}$$

Thus we have $?_j = \check{H}^j(\underline{H}_Y^0)$ $(j = 1, 2)$.

By Proposition 4.2, if $q > 0$ and X/Y is of degree p-power and $p | \deg(\pi_2)$, $\check{H}^q(\underline{H}_Y^0)$ is a p-group, killed by U^M for $M \gg 0$. Taking $(X, Y, U)_{/S}$ to be $(X_{s/\mathbb{Q}}, X_{s/\mathbb{Q}}^r, U(p))_{/\mathbb{Q}}$ for $s > r \geq 1$, we get for the projection $\pi : X_s \to X_s^r$

Corollary 4.5. *Let F be a number field or a finite extension of $\mathbb{Q}_l$ for a prime l. Then we have*

(u) $\pi^* : J_{s/\mathbb{Q}}^r(F) \to \check{H}^0(X_s/X_s^r, J_{s/\mathbb{Q}}(F)) \overset{(*)}{=} J_{s/\mathbb{Q}}(F)[\gamma^{p^{r-\epsilon}} - 1]$ *is a $U(p)$-isomorphism,*

where $J_{s/\mathbb{Q}}(F)[\gamma^{p^{r-\epsilon}} - 1] = \mathrm{Ker}(\gamma^{p^{r-\epsilon}} - 1 : J_s(F) \to J_s(F))$.

Here the identity at $(*)$ follows from Lemma 4.4. The kernel $A \mapsto \mathrm{Ker}(\gamma^{p^{r-\epsilon}} - 1 : J_s(A) \to J_s(A))$ is an abelian fppf sheaf (as the category of abelian fppf sheaves is abelian and regarding a sheaf as a presheaf is a left exact functor), and it is represented by the scheme theoretic kernel $J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}} - 1]$

of the endomorphism $\gamma^{p^{r-\epsilon}}-1$ of $J_{s/\mathbb{Q}}$. From the exact sequence $0 \to J_s[\gamma^{p^{r-\epsilon}}-1] \to J_s \xrightarrow{\gamma^{p^{r-\epsilon}}-1} J_s$, we get another exact sequence

$$0 \to J_s[\gamma^{p^{r-\epsilon}}-1](F) \to J_s(F) \xrightarrow{\gamma^{p^{r-\epsilon}}-1} J_s(F).$$

Thus

$$J_{s/\mathbb{Q}}(F)[\gamma^{p^{r-\epsilon}}-1] = J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}}-1](F).$$

The above (u) combined with (u1) implies (u2) below:

(u2) $\pi^* : J_{r/\mathbb{Q}} \to J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}}-1] = \mathrm{Ker}(\gamma^{p^{r-\epsilon}}-1 : J_{s/\mathbb{Q}} \to J_{s/\mathbb{Q}})$ is a $U(p)$-isomorphism.

Actually we can reformulate these facts as

Lemma 4.6. *Then we have morphisms*

$$\iota^r_s : J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}}-1] \to J^r_{s/\mathbb{Q}} \quad \text{and} \quad \iota^{r,*}_s : J^r_{s/\mathbb{Q}} \to J_{s/\mathbb{Q}}/(\gamma^{p^{r-\epsilon}}-1)(J_{s/\mathbb{Q}})$$

satisfying the following commutative diagrams:

$$(4.4)\qquad \begin{array}{ccc} J^r_{s/\mathbb{Q}} & \xrightarrow{\pi^*} & J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}}-1] \\ \downarrow u & \swarrow \iota^r_s & \downarrow u'' \\ J^r_{s/\mathbb{Q}} & \xrightarrow{\pi^*} & J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}}-1], \end{array}$$

and

$$(4.5)\qquad \begin{array}{ccc} J^r_{s/\mathbb{Q}} & \xleftarrow{\pi_*} & J_{s/\mathbb{Q}}/(\gamma^{p^{r-\epsilon}}-1)(J_{s/\mathbb{Q}}) \\ \uparrow u^* & \nearrow \iota^{r,*}_s & \uparrow u''^* \\ J^r_{s/\mathbb{Q}} & \xleftarrow{\pi_*} & J_{s/\mathbb{Q}}/(\gamma^{p^{r-\epsilon}}-1)(J_{s/\mathbb{Q}}), \end{array}$$

where u and u'' are $U(p^{s-r}) = U(p)^{s-r}$ and u^* and u''^* are $U^*(p^{s-r}) = U^*(p)^{s-r}$. In particular, for an fppf extension $T_{/\mathbb{Q}}$, the evaluated map at T: $(J_{s/\mathbb{Q}}/(\gamma^{p^{r-\epsilon}}-1)(J_{s/\mathbb{Q}}))(T) \xrightarrow{\pi_*} J^r_s(T)$ (resp. $J^r_s(T) \xrightarrow{\pi^*} J_s[\gamma^{p^{r-\epsilon}}-1](T)$) is a $U^*(p)$-isomorphism (resp. $U(p)$-isomorphism).

Note here that the natural homomorphism:

$$\frac{J_s(T)}{(\gamma^{p^{r-\epsilon}}-1)(J_s(T))} \to (J_{s/\mathbb{Q}}/(\gamma^{p^{r-\epsilon}}-1)(J_{s/\mathbb{Q}}))(T)$$

may have non-trivial kernel and cokernel which may not be killed by a power of $U^*(p)$. In other words, the left-hand-side is an fppf presheaf (of T) and the right-hand-side is its sheafication. On the other hand, $T \mapsto J_s[\gamma^{p^{r-\epsilon}}-1](T)$ is already an fppf abelian sheaf; so, $J^r_s(T) \xrightarrow{\pi^*} J_s[\gamma^{p^{r-\epsilon}}-1](T)$ is a $U(p)$-isomorphism without ambiguity.

Proof. We first prove the assertion for π^*. We note that the category of groups schemes fppf over a base S is a full subcategory of the category of abelian fppf sheaves. We may regard $J^r_{s/\mathbb{Q}}$ and $J_s[\gamma^{p^{r-\epsilon}}-1]_{/\mathbb{Q}}$ as abelian fppf sheaves over $\mathbb{Q}$ in this proof. Since these sheaves are represented by (reduced) algebraic groups over $\mathbb{Q}$, we can check being $U(p)$-isomorphism by evaluating the sheaf at a field k of characteristic 0 (e.g., [EAI, Lemma 4.18]). By Proposition 4.2 (2) applied to $X = X_{s/k} = X_s \times_{\mathbb{Q}} k$ and $Y = X^r_{s/k}$ (with $S = \mathrm{Spec}(k)$ and $s \geq r$),

$$\mathcal{K} := \mathrm{Ker}(J^r_{s/\mathbb{Q}} \to J_{s/\mathbb{Q}}[\gamma^{p^{r-\epsilon}}-1])$$

is killed by $U(p)^{s-r}$ as $d = p^{s-r} = \deg(X_s/X^r_s)$. Thus we get

$$\mathcal{K} \subset \mathrm{Ker}(U(p)^{s-r} : J^r_{s/\mathbb{Q}} \to J^r_{s/\mathbb{Q}}).$$

Since the category of fppf abelian sheaves is an abelian category (because of the existence of the sheafication functor from presheaves to sheaves under fppf topology described in [ECH, §II.2]), the above inclusion implies the existence of ι^r_s with $\pi^* \circ \iota^r_s = U(p)^{s-r}$ as a morphism of abelian fppf sheaves. Since the category of group schemes fppf over a base S is a full subcategory of the category of abelian fppf sheaves, all morphisms appearing in the identity $\pi^* \circ \iota^r_s = U(p)^{s-r}$ are morphism of group schemes. This proves the assertion for π^*.

Note that the second assertion is the dual of the first; so, it can be proven reversing all the arrows and replacing $J_s[\gamma^{p^{r-\epsilon}} - 1]_{/\mathbb{Q}}$ (resp. π^*, $U(p)$) by the quotient $J_s/(\gamma^{p^{r-\epsilon}} - 1)J_s$ as fppf abelian sheaves (resp. π_*, $U^*(p)$). Since $J_s/(\gamma^{p^{r-\epsilon}} - 1)(J_s)$ and J^r_s are abelian schemes over $\mathbb{Q}$, the quotient abelian scheme $J_s/(\gamma^{p^{r-\epsilon}} - 1)(J_s)$ is the dual of $J_s[\gamma^{p^{r-\epsilon}} - 1]$ and $\iota^{r,*}_s$ is the dual of ι^r_s. □

By the second diagram of the above lemma, we get

(u*) $J_s/(\gamma^{p^{r-\epsilon}} - 1)(J_s)_{/\mathbb{Q}} \xrightarrow{\pi_*} J^r_{s/\mathbb{Q}}$ is a $U^*(p)$-isomorphism of abelian fppf sheaves.

As a summary, we have

Corollary 4.7. *Then the morphism $\pi : X_s \to X^r_s$ induces an isogeny*

$$\overline{\pi}_* : J_s/(\gamma^{p^{r-\epsilon}} - 1)(J_s)_{/\mathbb{Q}} \to J^r_{s/\mathbb{Q}}$$

whose kernel is killed by a sufficiently large power of $U^*(p)$, and the pull-back map π^* induces an isogeny $\underline{\pi}^* : J_s[\gamma^{p^{r-\epsilon}} - 1] \to J^r_s$ whose kernel is killed by a high power of $U(p)$. Moreover, for a finite extension F of $\mathbb{Q}$ or $\mathbb{Q}_l$ (for a prime l not necessarily equal to p), $\underline{\pi}^* : J_s[\gamma^{p^{r-\epsilon}} - 1](F) \to J^r_s(F)$ is a $U(p)$-isomorphism.

Proof. Let $C \subset \mathrm{Aut}(X_s)$ be the cyclic group generated by the action of $\gamma^{p^{r-\epsilon}}$. Then $X_{s/\overline{\mathbb{Q}}}/X^r_{s/\overline{\mathbb{Q}}}$ is an étale covering with Galois group C (even unramified at cusps). Thus $Lie(J^r_s) = H^1(X^r_s, \mathcal{O}_{X^r_s}) = H_0(C, H^1(X_s, \mathcal{O}_{X_s})) = H_0(C, Lie(J_s))$. This shows that $\overline{\pi}_*$ is an isogeny over $\overline{\mathbb{Q}}$ and hence over $\mathbb{Q}$, which is a $U^*(p)$-isomorphism by Lemma 4.6. By taking dual, $\underline{\pi}^*$ is also an isogeny, which is a $U(p)$-isomorphism even after evaluating the fppf sheaves at F by Lemma 4.6 and the remark following the lemma. This proves the corollary. □

Then we get

(u*2) $J_s/(\gamma^{p^{r-\epsilon}} - 1)(J_s)_{/\mathbb{Q}} \to J_{r/\mathbb{Q}}$ is a $U^*(p)$-isomorphism of abelian fppf sheaves.

We can prove (u*2) in a more elementary way. We describe the easier proof. Identify $J_s(\mathbb{C}) = H^1(X_s, \mathbf{T})$ whose Pontryagin dual is given by $H_1(X_s, \mathbb{Z})$. If $k = \mathbb{Q}$, we have the Pontryagin dual version of (u2):

$$H_1(X_r, \mathbb{Z}) \xleftarrow{\pi_*} H_1(X_s, \mathbb{Z})/(\gamma^{p^{r-\epsilon}} - 1)(H_1(X_s, \mathbb{Z})) \text{ is a } U^*(p)\text{-isomorphism.} \tag{4.6}$$

Since $J_{s,\mathbb{Q}}(\mathbb{C}) \cong H_1(X_s, \mathbb{R})/H_1(X_s, \mathbb{Z})$ as Lie groups, we get

$$J_r(\mathbb{C}) \xleftarrow{\pi_*} J_s(\mathbb{C})/(\gamma^{p^{r-\epsilon}} - 1)(J_s(\mathbb{C})) \text{ is a } U^*(p)\text{-isomorphism.} \tag{4.7}$$

This implies (u*2). By (4.7), writing $\overline{\mathbb{Q}}$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and taking algebraic points, we get

$$J_r(\overline{\mathbb{Q}}) \xleftarrow{\pi_*} (J_s/(\gamma^{p^{r-\epsilon}} - 1)(J_s))(\overline{\mathbb{Q}}) = J_s(\overline{\mathbb{Q}})/(\gamma^{p^{r-\epsilon}} - 1)(J_s(\overline{\mathbb{Q}})) \text{ is a } U^*(p)\text{-isomorphism.} \tag{4.8}$$

Remark 4.8. The $U(p)$-isomorphisms of Jacobians do not kill the part associated to finite slope Hecke eigenforms. Thus the above information includes not just the information of p-ordinary forms but also those of finite slope Hecke eigenforms.

5. Control of Λ-MW groups as fppf sheaves

Let k be either a number field in $\overline{\mathbb{Q}}$ or a finite extension of $\mathbb{Q}_l$ in $\overline{\mathbb{Q}}_l$ for a prime l. Write O_k (resp. W) for the (resp. l-adic) integer ring of k if k is a number field (resp. a finite extension of $\mathbb{Q}_l$). For an abelian variety $A_{/k}$, we have $\widehat{A}(\kappa) := \varprojlim A(\kappa)/p^n A(\kappa)$ for a finite field extension κ/k as in (2.2). A down-to-earth description of the value of $\widehat{A}(\kappa)$ is given by (S) just above (2.2).

We study $J_r(k)$ equipped with the topology $J_k(k)$ induced from k (so, it is discrete if k is a number field and is l-adic if k is a finite extension of $\mathbb{Q}_l$). The p-adic limits $e = \lim_{n\to\infty} U(p)^{n!}$ and

$e^* = \lim_{n\to\infty} U^*(p)^{n!}$ are well defined on $\widehat{J}_r(k)$. The Albanese functoriality gives rise to a projective system $\{\widehat{J}_s(k), \pi_{s,r,*}\}_s$ for the covering map $\pi_{s,r} : X_s \to X_r$ $(s > r)$, and we have

$$\widetilde{J}_\infty(k) = \varprojlim_r \widehat{J}_r(k) \quad \text{(with projective limit of } p\text{-profinite compact topology)}$$

on which the co-ordinary projector $e^* = \lim_{n\to\infty} U^*(p)^{n!}$ acts. As before, adding superscript or subscript "ord" (resp. "co-ord"), we indicate the image of e (resp. e^*) depending on the situation.

We study mainly in this paper the control theorems of the w-twisted version $\widehat{J}_\infty(k)^{\mathrm{ord}}$ (which we introduce in Section 6) of $\widetilde{J}_\infty(k)^{\text{co-ord}}$ under the action of Γ and Hecke operators, and we have studied $\check{J}^{\mathrm{ord}}_\infty(k)$ in [H15] in a similar way. Here the word "w-twisting" means modifying the transition maps by the Weil involution at each step. As fppf sheaves, we have an isomorphism $i : \widehat{J}_\infty(k)^{\mathrm{ord}} \cong \widetilde{J}_\infty(k)^{\text{co-ord}}$ but $i \circ T(n) = T^*(n) \circ i$ for all n. Hereafter, unless otherwise mentioned, once our fppf abelian sheaf is evaluated at k, all morphisms are continuous with respect to the topology defined above (and we do not mention continuity often).

From (u1), we get

(5.1) $\quad J_r(k) \xrightarrow{\pi^*} J_s^r(k)$ is a $U(p)$-isomorphism (for the projection $\pi : X_s^r \to X_r$).

The dual version (following from (u*1)) is

(5.2) $\quad J_s^r(k) \xrightarrow{\pi_*} J_r(k)$ is a $U^*(p)$-isomorphism, where π_* is the dual of π^*.

From (5.1) and (5.2), we get

Lemma 5.1. *For a field k as above, we have*

$$\pi_* : \widehat{J}_s^r(k)^{\text{co-ord}} \cong \widehat{J}_r(k)^{\text{co-ord}} \quad \textit{and } \pi^* : \widehat{J}_r(k)^{\mathrm{ord}} \cong \widehat{J}_s^r(k)^{\mathrm{ord}}$$

for all $0 < r < s$ with the projection $\pi : X_s^r \twoheadrightarrow X_r$.

From Corollary 4.7 (or Lemma 4.6 combined with (u*2) and (u2)), for any field k, we get

(I) $\pi^* : J_r(k) \to J_s[\gamma^{p^{r-\epsilon}} - 1](k)$ is a $U(p)$-isomorphism, and obviously, $\pi^* : J_r \to J_s[\gamma^{p^{r-\epsilon}} - 1]$ is a $U(p)$-isomorphism of abelian fppf sheaves.

(P) $\pi_* : J_r \to J_s/(\gamma^{p^{r-\epsilon}} - 1)J_s$ is a $U^*(p)$-isomorphism of fppf abelian sheaves.

Note that (P) does not mean that $\frac{\widehat{J}_s(k)}{(\gamma^{p^{r-\epsilon}}-1)(\widehat{J}_s(k))} \to \widehat{J}_r(k)$ is a $U^*(p)$-isomorphism (as the sheaf quotient $J_s/(\gamma^{p^{r-\epsilon}} - 1)J_s$ and the corresponding presheaf quotient could be different).

We now claim

Lemma 5.2. *For integers $0 < r < s$, we have isomorphisms of fppf abelian sheaves*

$$\pi^* : \widehat{J}_r^{\mathrm{ord}} \cong \widehat{J}_s[\gamma^{p^{r-\epsilon}} - 1]^{\mathrm{ord}} \quad \textit{and } \pi_* : \left(\widehat{\frac{J_s}{(\gamma^{p^{r-\epsilon}} - 1)J_s}}\right)^{\text{co-ord}} \cong \widehat{J}_r^{\text{co-ord}}.$$

The first isomorphism π^* induces an isomorphism: $\widehat{J}_r^{\mathrm{ord}}(T) \cong \widehat{J}_s[\gamma^{p^{r-\epsilon}} - 1]^{\mathrm{ord}}(T)$ for any fppf extension $T_{/k}$ but the morphism induced by the second one: $\frac{\widehat{J}_s(T)^{\text{co-ord}}}{(\gamma^{p^{r-\epsilon}}-1)(\widehat{J}_s(T))^{\text{co-ord}}} \to \widehat{J}_r(T)^{\text{co-ord}}$ may not be an isomorphism.

Proof. By (I) above, $\widehat{J}_r^{\mathrm{ord}} \cong \widehat{A}^{\mathrm{ord}}$ for the abelian variety $A = J_s[\gamma^{p^{r-\epsilon}} - 1]$ and $\widehat{A}$ as in (S) above (2.2). We consider the following exact sequence

$$0 \to A \to J_s \xrightarrow{\gamma^{p^{r-\epsilon}}-1} J_s.$$

This produces another exact sequence $0 \to \widehat{A} \to \widehat{J}_s \xrightarrow{\gamma^{p^{r-\epsilon}}-1} \widehat{J}_s$; so, we get $\widehat{A} \cong \widehat{J}_s[\gamma^{p^{r-\epsilon}} - 1]$. Taking ordinary part and combining with the identity: $\widehat{J}_r^{\mathrm{ord}} \cong \widehat{A}^{\mathrm{ord}}$, we conclude $\widehat{J}_r^{\mathrm{ord}} \cong \widehat{J}_s[\gamma^{p^{r-\epsilon}} - 1]^{\mathrm{ord}}$. This holds true after evaluation at T as the presheaf-kernel of a sheaf morphism is still a sheaf. The second assertion is the dual of the first. □

Passing to the limit, Lemmas 5.1 and 5.2 tells us

Theorem 5.3. *Let k be either a number field or a finite extension of $\mathbb{Q}_l$. Then we have isomorphisms of fppf abelian sheaves over k:*

(a) $J_\infty^{\mathrm{ord}}[\gamma^{p^{r-\epsilon}}-1] \cong \widehat{J}_r^{\mathrm{ord}}$;

(b) $(\widetilde{J}_\infty/(\gamma^{p^{r-\epsilon}}-1)(\widetilde{J}_\infty))^{\text{co-ord}} \cong \widehat{J}_r^{\text{co-ord}}$

where we put $\widetilde{J}_\infty/(\gamma^{p^{r-\epsilon}}-1)(\widetilde{J}_\infty))^{\text{co-ord}} := \varprojlim_s \widehat{J_s/(\gamma^{p^{r-\epsilon}}-1)(J_s)}^{\text{co-ord}}$ *as an fppf sheaf.*

Proof. The assertion (b) is just the projective limit of the corresponding statement in Lemma 5.2.

We prove (a). Since injective limit always preserves exact sequences, we have

$$0 \to \widehat{J}_r(k)^{\mathrm{ord}} \to \varinjlim_s \widehat{J}_s(k)^{\mathrm{ord}} \xrightarrow{\gamma^{p^{r-\epsilon}}-1} \varinjlim_s \widehat{J}_s(k)^{\mathrm{ord}}$$

is exact, showing (a). □

See [H15, Proposition 6.4] for a control result similar to (a) for $\check{J}_\infty^{\mathrm{ord}}$.

Remark 5.4. *As is clear from the warning after* (P)*, the isomorphism* (b) *does* **not** *mean that*

$$\varprojlim_s \left\{ \frac{\widehat{J}_s(T)}{(\gamma^{p^{r-\epsilon}}-1)(\widehat{J}_s(T))} \right\}^{\text{co-ord}} \to \widehat{J}_r(T)^{\text{co-ord}}$$

for each fppf extension $T_{/k}$ is an isomorphism. The kernel and the cokernel of this map will be studied in Section 9.

6. Sheaves associated to modular Jacobians

We fix an element $\zeta \in \mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}(\overline{\mathbb{Q}})$; so, ζ is a coherent sequence of generators ζ_{p^n} of $\mu_{p^n}(\overline{\mathbb{Q}})$ (i.e., $\zeta_{p^{n+1}}^p = \zeta_{p^n}$). We also fix a generator ζ_N of $\mu_N(\overline{\mathbb{Q}})$, and put $\zeta_{Np^r} := \zeta_N\zeta_{p^r}$. Identify the étale group scheme $\mathbb{Z}/Np^n\mathbb{Z}_{/\mathbb{Q}[\zeta_N,\zeta_{p^n}]}$ with μ_{Np^n} by sending $m \in \mathbb{Z}$ to $\zeta_{Np^n}^m$. Then for a couple $(E, \phi_{Np^r} : \mu_{Np^r} \hookrightarrow E)_{/K}$ over a $\mathbb{Q}[\mu_{p^r}]$-algebra K, let $\phi^* : E[Np^r] \twoheadrightarrow \mathbb{Z}/Np^r\mathbb{Z}$ be the Cartier dual of ϕ_{Np^r}. Then ϕ^* induces $E[Np^r]/\operatorname{Im}(\phi_{Np^r}) \cong \mathbb{Z}/Np^r\mathbb{Z}$. Define $i : \mathbb{Z}/p^r\mathbb{Z} \cong (E/\operatorname{Im}(\phi_{Np^r}))[Np^r]$ by the inverse of ϕ^*. Then we define $\varphi_{Np^r} : \mu_{Np^r} \hookrightarrow E/\operatorname{Im}(\phi_{Np^r})$ by $\varphi_{Np^r} : \mu_{Np^r} \cong \mathbb{Z}/Np^r\mathbb{Z} \xrightarrow{i} (E/\operatorname{Im}(\phi_{Np^r}])[p^r] \subset E/\operatorname{Im}(\phi_{Np^r})$. This induces an automorphism w_r of X_r defined over $\mathbb{Q}[\mu_{Np^r}]$, which in turn induces an automorphism w_r of $J_{r/\mathbb{Q}[\zeta_{Np^r}]})$. We have the following well known commutative diagram (e.g., [MFM, Section 4.6]):

$$\begin{array}{ccc} J_r & \xrightarrow{T(n)} & J_r \\ {\scriptstyle w_r}\downarrow\wr & & {\scriptstyle w_r}\downarrow\wr \\ J_r & \xrightarrow[T^*(n)]{} & J_r. \end{array}$$

Let $P \in \operatorname{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ be an arithmetic point of weight 2. Then we have a p-stabilized Hecke eigenform form f_P associated to P; i.e., $f_P|T(n) = P(T(n))f_P$ for all n. Then $f_P^* = w_r(f_P)$ is the dual common eigenform of $T^*(n)$. If f_P is new at every prime $l|pN$, f_P^* is a constant multiple of the complex conjugate f_P^c of f_P (but otherwise, it is different).

We then define as described in (S) in Section 2 an fppf abelian sheaf $\widehat{X}$ for any abelian variety quotient or subgroup variety X of $J_{s/k}$ over the fppf site over $k = \mathbb{Q}$ and $\mathbb{Q}_l$ (note here the explicit value of $\widehat{J}_s$ depends on k as in (S)).

Pick an automorphism $\sigma \in \operatorname{Gal}(\mathbb{Q}(\mu_{Np^r})/\mathbb{Q})$ with $\zeta_{Np^r}^\sigma = \zeta_{Np^r}^z$ for $z \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$. Since w_r^σ is defined with respect to $\zeta_{Np^r}^\sigma = \zeta_{Np^r}^z$, we find $w_r^\sigma = \langle z\rangle \circ w_r = w_r \circ \langle z\rangle^{-1}$ (see [MW86, page 237] and [MW84, 2.5.6]). Here $\langle z\rangle$ is the image of z in $(\mathbb{Z}/Np^r\mathbb{Z})^\times = \operatorname{Gal}(X_r/X_0(Np^r))$. Let $\pi_{s,r,*} : J_s \to J_r$ for $s > r$ be the morphism induced by the covering map $X_s \twoheadrightarrow X_r$ through Albanese functoriality. Then we define $\pi_s^r = w_r \circ \pi_{s,r,*} \circ w_s$. Then $(\pi_s^r)^\sigma = w_r\langle z^{-1}\rangle\pi_{s,r,*}\langle z\rangle w_s = \pi_s^r$ for all $\sigma \in \operatorname{Gal}(\mathbb{Q}(\mu_{Np^s})/\mathbb{Q})$; thus, π_s^r is well defined over $\mathbb{Q}$, and satisfies $T(n) \circ \pi_s^r = \pi_s^r \circ T(n)$ for all n prime to Np and $U(q) \circ \pi_s^r = \pi_s^r \circ U(q)$ for all $q|Np$. Since $w_r^2 = 1$, by this w-twisting, the projective system $\{J_s, \pi_{s,r,*}\}$ equivariant under $T^*(n)$ is transformed into the isomorphic projective system $\{J_s, \pi^r_s\}_{s>r}$ (of abelian varieties defined over $\mathbb{Q}$) which is Hecke equivariant (i.e., $T(n)$ and $U(l)$-equivariant). Thus what we proved for the co-ordinary part of the projective system $\{\widehat{J}_s, \pi_{s,r,*}\}$ is valid for the ordinary part of the projective system $\{\widehat{J}_s, \pi^r_s\}_{s>r}$. If X_s is either an algebraic subgroup or an abelian variety quotient of J_s and π^r_s produces a projective system $\{X_s\}_s$ we define $\widehat{X}_\infty := \varprojlim_s \widehat{X}_s(R)$ for an fppf extension R of $k = \mathbb{Q}, \mathbb{Q}_l$ (again the definition of $\widehat{X}_s$ and hence $\widehat{X}_\infty$ depends on k). For each ind-object $R = \varinjlim_i R_i$ of fppf, smooth or étale algebras R_i/k, we define $\widehat{X}_\infty(R) = \varinjlim_i \widehat{X}_\infty(R_i)$.

Lemma 6.1. *Let K/k be the Galois extension as in* Section 2. *Then the* $\mathrm{Gal}(K/k)$*-action on $\widehat{X}_\infty(K)$ is continuous under the discrete topology on $\widehat{X}_\infty(K)$. In particular, the Galois cohomology group $H^q(\widehat{X}_\infty(K)) := H^q(\mathrm{Gal}(K/\kappa), \widehat{X}_\infty(K))$ for $q > 0$ is a torsion $\mathbb{Z}_p$-module for any intermediate extension $K/\kappa/k$.*

Proof. By definition, $\widehat{X}_\infty(K) = \bigcup_{K/F/k} \widehat{X}_\infty(F)$, and $\widehat{X}_\infty(F) \subset H^0(\mathrm{Gal}(K/F), \widehat{X}_\infty(K))$ for all finite intermediate extension $K/F/k$. Thus $\widehat{X}_\infty(K) = \varinjlim_F H^0(\mathrm{Gal}(K/F), \widehat{X}_\infty(K))$, which implies the continuity of the action under the discrete topology. Then the torsion property follows from [MFG, Corollary 4.26]. □

Let $\iota : C_{r/\mathbb{Q}} \subset J_{r/\mathbb{Q}}$ be an abelian subvariety stable under Hecke operators (including $U(l)$ for $l|Np$) and w_r and ${}^t\iota : J_{r/\mathbb{Q}} \twoheadrightarrow {}^tC_{r/\mathbb{Q}}$ be the dual abelian quotient. We then define $\pi : J_r \twoheadrightarrow D_r$ by $D_r := {}^tC_r$ and $\pi = {}^tw_r \circ {}^t\iota_r \circ w_r$ for the map ${}^tw_r \in \mathrm{Aut}({}^tC_{r/\mathbb{Q}[\mu_{p^r}]})$ dual to $w_r \in \mathrm{Aut}(C_{r/\mathbb{Q}[\mu_{p^r}]})$. Again π is defined over $\mathbb{Q}$. Then ι and π are Hecke equivariant. Let $\iota_s : C_s := \pi^*_{s,r}(C) \subset J_s$ for $s > r$ and D_s be the quotient abelian variety of J_s defined in the same way taking C_s in place of C_r (and replacing r by s). Put $\pi_s : J_s \twoheadrightarrow D_s$ which is Hecke equivariant.

Since the two morphisms $J_r \to J^r_s$ and $J^r_s \to J_s[\gamma^{p^{r-\epsilon}} - 1]$ (Picard functoriality) are $U(p)$-isomorphism of fppf abelian sheaves by (u1) and Corollary 4.5, we get the following two isomorphisms of fppf abelian sheaves for $s > r > 0$:

$$C_r[p^\infty]^{\mathrm{ord}} \xrightarrow[\pi^*_{r,s}]{\sim} C_s[p^\infty]^{\mathrm{ord}} \text{ and } \widehat{C}^{\mathrm{ord}}_r \xrightarrow[\pi^*_{r,s}]{\sim} \widehat{C}^{\mathrm{ord}}_s, \tag{6.1}$$

since $\widehat{C}^{\mathrm{ord}}_s$ is the isomorphic image of $\widehat{C}^{\mathrm{ord}}_r \subset \widehat{J}_r$ in $\widehat{J}_s[\gamma^{p^{r-\epsilon}} - 1]$. By w-twisted Cartier duality [H14, §4], we have

$$D_s[p^\infty]^{\mathrm{ord}} \xrightarrow[\pi^r_s]{\sim} D_r[p^\infty]^{\mathrm{ord}}. \tag{6.2}$$

Thus by Kummer sequence in Lemma 2.1, we have the following commutative diagram

$$\begin{array}{ccc} \widehat{D}^{\mathrm{ord}}_s(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z} = (D_s(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z})^{\mathrm{ord}} & \xrightarrow[\hookrightarrow]{} & H^1(D_s[p^m]^{\mathrm{ord}}) \\ \pi^r_s \downarrow & & \wr \downarrow (6.2) \\ \widehat{D}^{\mathrm{ord}}_r(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z} = (D_r(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z})^{\mathrm{ord}} & \xrightarrow[\hookrightarrow]{} & H^1(D_r[p^m]^{\mathrm{ord}}) \end{array}$$

This shows

$$\widehat{D}^{\mathrm{ord}}_s(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z} \cong \widehat{D}^{\mathrm{ord}}_r(\kappa) \otimes \mathbb{Z}/p^m\mathbb{Z}.$$

Passing to the limit, we get

$$\widehat{D}^{\mathrm{ord}}_s \xrightarrow[\pi^r_s]{\sim} \widehat{D}^{\mathrm{ord}}_r \text{ and } (D_s \otimes_{\mathbb{Z}} \mathbb{T}_p)^{\mathrm{ord}} \xrightarrow[\pi^r_s]{\sim} (D_r \otimes_{\mathbb{Z}} \mathbb{T}_p)^{\mathrm{ord}} \tag{6.3}$$

as fppf abelian sheaves. In short, we get

Lemma 6.2. *Suppose that κ is a field extension of finite type of either a number field or a finite extension of $\mathbb{Q}_l$. Then we have the following isomorphism*

$$\widehat{C}_r(\kappa)^{\mathrm{ord}} \xrightarrow[\pi^*_{s,r}]{\sim} \widehat{C}_s(\kappa)^{\mathrm{ord}} \quad \textit{and} \quad \widehat{D}_s(\kappa)^{\mathrm{ord}} \xrightarrow[\pi^r_s]{\sim} \widehat{D}_r(\kappa)^{\mathrm{ord}}$$

for all $s > r$ including $s = \infty$.

By computation, $\pi^r_s \circ \pi^*_{r,s} = p^{s-r}U(p^{s-r})$. To see this, as Hecke operators coming from Γ_s-coset operations, $\pi^*_{r,s} = [\Gamma_s]$ (restriction map) and $\pi_{r,s,*} = [\Gamma_r]$ (trace operator for Γ_r/Γ_s). Thus we have (6.4)

$$\pi^r_s \circ \pi^*_{r,s}(x) = x|[\Gamma_s]\cdot w_s\cdot[\Gamma_r]\cdot w_r = x|[\Gamma_s]\cdot[w_s w_r]\cdot[\Gamma_r] = x|[\Gamma^r_s : \Gamma_s][\Gamma^r_s \begin{pmatrix} 1 & 0 \\ 0 & p^{s-r}\end{pmatrix} \Gamma_r] = p^{s-r}(x|U(p^{s-r})).$$

Corollary 6.3. *We have the following two commutative diagrams for $s' > s$*

$$\begin{array}{ccc} \widehat{C}^{\mathrm{ord}}_{s'} & \xleftarrow[\pi^*_{s,s'}]{\sim} & \widehat{C}^{\mathrm{ord}}_{s} \\ {\scriptstyle \pi^s_{s'}}\downarrow & & \downarrow{\scriptstyle p^{s'-s}U(p)^{s'-s}} \\ \widehat{C}^{\mathrm{ord}}_{s} & = & \widehat{C}^{\mathrm{ord}}_{s}. \end{array}$$

and

$$\begin{array}{ccc} \widehat{D}^{\mathrm{ord}}_{s'} & \xrightarrow[\pi^s_{s'}]{\sim} & \widehat{D}^{\mathrm{ord}}_{s} \\ {\scriptstyle \pi^*_{s,s'}}\uparrow & & \uparrow{\scriptstyle p^{s'-s}U(p)^{s'-s}} \\ \widehat{D}^{\mathrm{ord}}_{s} & = & \widehat{D}^{\mathrm{ord}}_{s}. \end{array}$$

Proof. By $\pi^*_{r,s}$ (resp. π^r_s), we identify $\widehat{C}^{\mathrm{ord}}_s$ with $\widehat{C}^{\mathrm{ord}}_r$ (resp. $\widehat{D}^{\mathrm{ord}}_s$ with $\widehat{D}^{\mathrm{ord}}_r$) as in Lemma 6.2. Then the above two diagrams follow from (6.4). □

By (6.4), we have exact sequences

$$(6.5)\quad \begin{aligned} 0 \to & C_s[p^{s-r}]^{\mathrm{ord}} \to C_s[p^\infty]^{\mathrm{ord}} \xrightarrow{\pi^r_s} C_r[p^\infty]^{\mathrm{ord}} \to 0, \\ 0 \to & D_r[p^{s-r}]^{\mathrm{ord}} \to D_r[p^\infty]^{\mathrm{ord}} \xrightarrow{\pi^*_{r,s}} D_s[p^\infty]^{\mathrm{ord}} \to 0. \end{aligned}$$

Applying (2.1) to the exact sequence $\mathcal{K}^r_s(K) \hookrightarrow C_s(K) \twoheadrightarrow C_r(K)$ for $\mathcal{K}^r_s(K) = \mathrm{Ker}(\pi^r_s)(K)$ and $\mathcal{K}_{r,s}(K) \hookrightarrow C_r(K) \twoheadrightarrow D_s(K)$ for $\mathcal{K}_{r,s} = \mathrm{Ker}(\pi^*_{r,s})$, we get the following exact sequence of fppf abelian sheaves:

$$\begin{aligned} 0 \to & \widehat{\mathcal{K}}^r_s \to \widehat{C}_s \xrightarrow{\pi^r_s} \widehat{C}_r \to 0, \\ 0 \to & \widehat{\mathcal{K}}_{r,s} \to \widehat{D}_r \xrightarrow{\pi^*_{r,s}} \widehat{D}_s \to 0. \end{aligned}$$

Taking the ordinary part, we confirm exactness of

$$(6.6)\quad \begin{aligned} 0 \to & C_s[p^{s-r}]^{\mathrm{ord}} \to \widehat{C}^{\mathrm{ord}}_s \xrightarrow{\pi^r_s} \widehat{C}^{\mathrm{ord}}_r \to 0, \\ 0 \to & D_r[p^{s-r}]^{\mathrm{ord}} \to \widehat{D}^{\mathrm{ord}}_r \xrightarrow{\pi^*_{r,s}} \widehat{D}^{\mathrm{ord}}_s \to 0. \end{aligned}$$

Write $H^1(X) = H^1(\mathrm{Gal}(K/\kappa), X)$ for an intermediate extension $K/\kappa/k$ and $\mathrm{Gal}(K/k)$-module X and $H^1_?(X) = H^1_?(\mathrm{Spec}(\kappa), X)$ for a smooth/fppf extension for ? = sm or fppf. Then taking the p-adic completion, we get the following exact sequences as parts of the long exact sequences associated to (6.6)

$$(6.7)\quad \begin{aligned} 0 \to & C_s[p^{s-r}]^{\mathrm{ord}}(\kappa) \to \widehat{C}^{\mathrm{ord}}_s(\kappa) \xrightarrow[\pi^r_s]{} \widehat{C}^{\mathrm{ord}}_r(\kappa) \to H^1_?(C_s[p^{s-r}]^{\mathrm{ord}}), \\ 0 \to & D_r[p^{s-r}]^{\mathrm{ord}}(\kappa) \to \widehat{D}^{\mathrm{ord}}_r(\kappa) \xrightarrow[\pi^*_{r,s}]{} \widehat{D}^{\mathrm{ord}}_s(\kappa) \to H^1_?(D_r[p^{s-r}]^{\mathrm{ord}}) \end{aligned}$$

for ? = fppf, sm (cohomology under smooth topology) or nothing (i.e., Galois cohomology equivalent to étale cohomology in this case). Here if ? = fppf, κ/k is an extension of finite type, if ? = sm, κ/k is a smooth extension of finite type, and if ? is nothing, $K/\kappa/k$ is an intermediate field.

By Lemma 6.2, we can rewrite the first exact sequence of (6.5) as

$$(6.8)\quad 0 \to C_r[p^{s-r}]^{\mathrm{ord}}(\kappa) \xrightarrow{\pi^*_{r,s}} \widehat{C}^{\mathrm{ord}}_s(\kappa) \xrightarrow{\pi^r_s} \widehat{C}^{\mathrm{ord}}_r(\kappa) \to H^1_?(C_r[p^{s-r}]^{\mathrm{ord}}).$$

This (combined with Corollary 6.3) induces the corresponding diagram for H^1, for any extension κ/k inside K,

$$\begin{array}{ccccc} H^1(C_s[p^{s'-r}]^{\mathrm{ord}}) & \xleftarrow[\pi^*_{r,s'}]{\sim} & H^1(C_r[p^{s'-r}]^{\mathrm{ord}}) & \hookleftarrow & \left(\frac{C_r(\kappa)}{p^{s'-r}C_r(\kappa)}\right)^{\mathrm{ord}} \\ \pi^s_{s'}\downarrow & & \downarrow p^{s'-s}U(p)^{s'-s} & & \downarrow p^{s'-s}U(p)^{s'-s} \\ H^1(C_s[p^{s-r}]^{\mathrm{ord}}) & \xleftarrow[\pi^*_{r,s}]{\sim} & H^1(C_r[p^{s-r}]^{\mathrm{ord}}) & \hookleftarrow & \left(\frac{C_r(\kappa)}{p^{s'-r}C_r(\kappa)}\right)^{\mathrm{ord}} . \end{array}$$

The right square is the result of Kummer theory for C_r. Passing to the projective limit with respect to s, we get a sequence

$$0 \to \varprojlim_s C_r[p^{s-r}]^{\mathrm{ord}}(\kappa) \xrightarrow{\pi^*_{r,s}} \varprojlim_s \widehat{C}^{\mathrm{ord}}_s(\kappa) \xrightarrow{\pi^r_s} \widehat{C}^{\mathrm{ord}}_r(\kappa) \to \varprojlim_s H^1(C_r[p^{s-r}]^{\mathrm{ord}}) \tag{6.9}$$

which is exact at left three terms up to the term $\widehat{C}^{\mathrm{ord}}_r(\kappa)$.

Proposition 6.4. *Let k be a finite extension field of $\mathbb{Q}$ or $\mathbb{Q}_l$ for a prime l. Assume* (2.2) *for $\kappa_{/k}$. Then we have the following identity*

$$\widehat{C}_\infty(\kappa)^{\mathrm{ord}} = \varprojlim_s \widehat{C}_s(\kappa)^{\mathrm{ord}} \cong \varprojlim_s C_r[p^{s-r}](\kappa)^{\mathrm{ord}} = 0$$

and an exact sequence for K/k as in Section 2*:*

$$\begin{aligned} 0 \to & T_pC^{\mathrm{ord}}_r \to \varprojlim_s \widehat{C}_s(K)^{\mathrm{ord}} \to \widehat{C}_r(K)^{\mathrm{ord}} \to 0 \\ 0 \to & T_pC^{\mathrm{ord}}_r \to \varprojlim_s C_s[p^\infty](K)^{\mathrm{ord}} \to C_r[p^\infty](K)^{\mathrm{ord}} \to 0. \end{aligned}$$

In the last sequence, we have $\varprojlim_s C_s[p^\infty](K)^{\mathrm{ord}} \cong T_pC^{\mathrm{ord}}_r \otimes_{\mathbb{Z}} \mathbb{Q}$. By the first identity, $\widehat{C}^{\mathrm{ord}}_\infty$ as a smooth (resp. étale) sheaf vanishes if k is a number field or a local field with residual characteristic $l \neq p$ (resp. a p-adic field).

Proof. By (6.9), we get a sequence which is exact at the first three left terms (up to the term $\widehat{C}^{\mathrm{ord}}_r(\kappa)$):

$$0 \to \varprojlim_s C_r[p^{s-r}](\kappa)^{\mathrm{ord}} \to \widehat{C}_\infty(\kappa)^{\mathrm{ord}} \xrightarrow[\pi^r_s]{} \widehat{C}_r(\kappa)^{\mathrm{ord}} \xrightarrow{\delta} \varprojlim_s H^1(C_s[p^{s-r}]^{\mathrm{ord}}).$$

Since δ is injective by Lemma 2.1 under (2.2), we get the first two identities. The vanishing of $\varprojlim_s C_r[p^{s-r}](\kappa)^{\mathrm{ord}}$ follows because $C_r[p^\infty]^{\mathrm{ord}}(\kappa)$ is a finite p-torsion module if κ/k is an extension of finite type.

If $\kappa = K$, we may again pass to the limit of the first exact sequence of (6.6) again noting $C_s[p^{s-r}](K)^{\mathrm{ord}} \cong C_r[p^{s-r}](K)^{\mathrm{ord}}$. The limit keeps exactness (as $\{C_r[p^{s-r}](K)\}_s$ is a surjective projective system), and we get the following exact sequence

$$0 \to TC_r[p^\infty](K)^{\mathrm{ord}} \to \varprojlim_s \widehat{C}_s(K)^{\mathrm{ord}} \xrightarrow[\pi^r_s]{} \widehat{C}_r(K)^{\mathrm{ord}} \to 0.$$

The divisible version can be proven taking the limit of (6.5). Since $C_r[p^\infty](K)^{\mathrm{ord}}$ is p-divisible and the projective system of the exact sequences $0 \to C_r[p](K)^{\mathrm{ord}} \to C_r[p^\infty](K)^{\mathrm{ord}} \xrightarrow{x \mapsto px} C_r[p^\infty](K)^{\mathrm{ord}} \to 0$ by the transition map $x \mapsto p^nU(p^n)(x)$ satisfies the Mittag–Leffler condition (as $C_r[p](K)^{\mathrm{ord}}$ is finite), $\varprojlim_s C_s[p^\infty](K)^{\mathrm{ord}}$ is a p-divisible module. Thus by the exact sequence, we have $T_pC^{\mathrm{ord}}_r \otimes_{\mathbb{Z}} \mathbb{Q} \subset \varprojlim_s C_s[p^\infty](K)^{\mathrm{ord}}$, which implies

$$T_pC^{\mathrm{ord}}_r \otimes_{\mathbb{Z}} \mathbb{Q} \cong \varprojlim_s C_s[p^\infty](K)^{\mathrm{ord}}$$

as $T_pC^{\mathrm{ord}}_r \otimes_{\mathbb{Z}} \mathbb{Q}/T_pC_r \cong C_r[p^\infty]^{\mathrm{ord}}(K)$. □

We insert here Shimura's definition of his abelian subvariety [IAT, Theorem 7.14] and abelian variety quotient [Sh73] of J_s associated to a member f_P of a p-adic analytic family. Shimura mainly considered these abelian varieties associated to a primitive Hecke eigenform. Since we need those associated to old Hecke eigenforms, we give some details.

Let $P \in \mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ be an arithmetic point of weight 2. Then we have a p-stabilized Hecke eigenform form f_P associated to P; i.e., $f_P|T(n) = P(T(n))f_P$ for all n (e.g., [GME, Section 3.2]). Then $f^*_P = w_r(f_P)$ is the dual common eigenform of $T^*(n)$. If f_P is new at every prime $l|pN$, f^*_P is a constant multiple of the complex conjugate f^c_P of f_P (but otherwise, they are different). Shimura's abelian subvariety A_P (associated to f_P) is defined to be the identity connected component of $\bigcap_{\alpha\in P} J_r[\alpha]$ regarding P as a prime ideal of $h_r(\mathbb{Z})$.

The Rosati involution (induced by the canonical polarization) brings $h_r(\mathbb{Z})$ to $h^*_r(\mathbb{Z}) \subset \mathrm{End}(J_{r/\mathbb{Q}})$ isomorphically, and $\mathbf{h}$ acts on $\widehat{J}_\infty$ (resp. $\widetilde{J}_\infty$) through the identity $T(n) \mapsto T(n)$ (resp. through $T(n) \mapsto T^*(n)$). Let $f^*_P|T^*(n) = P(T(n))f^*_P$, and regard P as an algebra homomorphism $P^* : h^*_r(\mathbb{Z}) \to \overline{\mathbb{Q}}$ (so, $P^*(T^*(n)) = P(T(n))$). Identify P^* with the prime ideal $\mathrm{Ker}(P^*)$, and define A^*_P to be the identity connected component of $J_r[P^*] := \bigcap_{\alpha\in P^*} J_r[\alpha]$. Then $A_P \cong A^*_P$ by w_r over $\mathbb{Q}(\mu_{Np^r})$.

Assume that $r = r(P)$ is the minimal exponent of p in the level of f_P. For $s > r$, we write A_s (resp. A^*_s) for the abelian variety associated to f_P regarded as an old form of level p^s (resp. $w_s(f_P)$). In other words, regarding P^* as an ideal of $h^*_s(\mathbb{Z})$ via the projection $h^*_s(\mathbb{Z}) \twoheadrightarrow h^*_r(\mathbb{Z})$, we define A^*_s by the identity connected component of $J_s[P^*]$. The Albanese functoriality $\pi_* : J_s \twoheadrightarrow J_r$ induces an isogeny $A^*_s \twoheadrightarrow A^*_r = A^*_P$. Similarly the Picard functoriality $\pi^* : J_r \to J_s$ induces an isogeny $A_P = A_r \twoheadrightarrow A_s$. Since f^*_P is the complex conjugate of f_P (assuming that f_P is new), $A^*_P = A_P$ inside J_r (for $r = r(P)$). Since $w_s : A_{s/\mathbb{Q}[\zeta_{Np^s}]} \cong A^*_{s/\mathbb{Q}[\zeta_{Np^s}]}$ and A_s and A^*_s are isogenous to A_P over $\mathbb{Q}$, A_s and A^*_s are isomorphic over $\mathbb{Q}$. Consider the dual quotient $J_s \twoheadrightarrow B_s$ (resp. $J_s \twoheadrightarrow B^*_s$) of $A^*_s \hookrightarrow J_s$ (resp. $A_s \hookrightarrow J_s$). In the same manner as above, B_s and B^*_s are isomorphic over $\mathbb{Q}$. Then B_s (resp. B^*_s) is stable under $T(n)$ and $U(p)$ (resp. $T^*(n)$ and $U^*(p)$) and $\Omega_{B_s/\mathbb{C}}$ (resp. $\Omega_{B^*_s/\mathbb{C}}$) is spanned by $f^\sigma_P dz$ (resp. $g^\sigma_P dz$ for $g_P = w_s(f_P)$) for σ running over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We mainly apply Corollary 6.3 and Proposition 6.4 taking C_s (resp. D_s) to be A_s (resp. B_s).

7. Abelian factors of modular Jacobians

Let k be a finite extension of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}$ or a finite extension of $\mathbb{Q}_p$ over $\overline{\mathbb{Q}}_p$. We study the control theorem for $\widehat{J}_s(k)$ which is not covered in [H15].

Let A_r be a group subscheme of J_r proper over k; so, A_r is an extension of an abelian scheme $A^\circ_{r/\mathbb{Q}}$ by a finite étale group. Write A_s $(s \geq r)$ for the image of A_r in J_s under the morphism $\pi^* : J_r \to J_s$ given by Picard functoriality from the projection $\pi : X_s \to X_r$. Hereafter we assume

(A) We have $\alpha \in \mathbf{h}(N)$ such that $(\gamma^{p^{r-\epsilon}} - 1) = \alpha x$ with $x \in \mathbf{h}(N)$ and that $\mathbf{h}/(\alpha)$ is free of finite rank over $\mathbb{Z}_p$. Write α_s for the image in $\mathbf{h}_s$ $(s \geq r)$ and $\mathfrak{a}_s = (\alpha_s \mathbf{h}_s \oplus (1-e)h_s(\mathbb{Z}_p)) \cap h_s(\mathbb{Z})$ and put $A_s = J_s[\mathfrak{a}_s]$ and $B_s = J_s/\mathfrak{a}_s J_s$, where $\mathfrak{a}_s J_s$ is an abelian subvariety defined over $\mathbb{Q}$ of J_s with $\mathfrak{a}_s J_s(\overline{\mathbb{Q}}) = \sum_{a\in\mathfrak{a}_s} a(J_s(\overline{\mathbb{Q}})) \subset J_s(\overline{\mathbb{Q}})$.

Here for $s > s'$, coherency of α_s means the following commutative diagram:

$$\begin{array}{ccc} \widehat{J}^{\mathrm{ord}}_{s'} & \xrightarrow{\pi^*} & \widehat{J}^{\mathrm{ord}}_{s} \\ {\scriptstyle \alpha_{s'}}\downarrow & & \downarrow{\scriptstyle \alpha_s} \\ \widehat{J}^{\mathrm{ord}}_{s'} & \xrightarrow[\pi^*]{} & \widehat{J}^{\mathrm{ord}}_{s} \end{array}$$

which is equivalent (by the self-duality of J_s) to the commutativity of

$$\begin{array}{ccc} \widehat{J}^{\text{co-ord}}_{s} & \xrightarrow{\pi_*} & \widehat{J}^{\text{co-ord}}_{s'} \\ {\scriptstyle \alpha^*_{s}}\downarrow & & \downarrow{\scriptstyle \alpha^*_{s'}} \\ \widehat{J}^{\text{co-ord}}_{s} & \xrightarrow[\pi_*]{} & \widehat{J}^{\text{co-ord}}_{s'}. \end{array}$$

The following fact is proven in [H16, Lemma 5.1]:

Lemma 7.1. *Assume* (A). *Then we have* $\widehat{A}_s^{\mathrm{ord}} = \widehat{J}_s^{\mathrm{ord}}[\alpha_s]$ *and* $\widehat{A}_s^{\circ} = \widehat{A}_s$. *The identity connected component* A_s° $(s > r)$ *of* A_s *is the image of* A_r° *in* J_s *under the morphism* $\pi^* = \pi_{s,r}^* : J_r \to J_s$ *induced by Picard functoriality from the projection* $\pi = \pi_{s,r} : X_s \to X_r$ *and is* $\mathbb{Q}$*-isogenous to* B_s. *The morphism* $J_s \to B_s$ *factors through* $J_s \xrightarrow{\pi_s^r} J_r \to B_r$. *In addition, the sequence*

$$0 \to \widehat{A}_s^{\mathrm{ord}} \to \widehat{J}_s^{\mathrm{ord}} \xrightarrow{\alpha} \widehat{J}_s^{\mathrm{ord}} \xrightarrow{\rho_s} \widehat{B}_s^{\mathrm{ord}} \to 0 \quad \text{for } 0 < \epsilon \leq r \leq s < \infty$$

is an exact sequence of fppf sheaves.

This implies

Corollary 7.2. *Recall the finite set* S *of places made of prime factors of* Np *and* ∞. *Let* $R = k$ *if* k *is local, and let* R *be the* S*-integer ring of* k *(i.e., primes in* S *is inverted in* R*) if* k *is a number field. Then the sheaf* $\alpha_s(\widehat{J}_s^{\mathrm{ord}})$ *is a* p*-divisible étale/fppf sheaf over* $\mathrm{Spec}(R)$, *and its* p*-torsion part* $\alpha_s(\widehat{J}_s^{\mathrm{ord}})[p^\infty]$ *is a* p*-divisible Barsotti–Tate group over* R.

In particular, the Tate module $T_p\alpha(\widehat{J}_s^{\mathrm{ord}})$ is a well defined free $\mathbb{Z}_p$-module of finite rank for all $r \leq s < \infty$.

Proof. By the above lemma, the fppf sheaf $\alpha_s(\widehat{J}_s^{\mathrm{ord}}) = \mathrm{Ker}(\widehat{J}_s^{\mathrm{ord}} \xrightarrow{\rho_s} \widehat{B}_s^{\mathrm{ord}})$ fits into the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
A_s[p^\infty]^{\mathrm{ord}} & \xrightarrow{\hookrightarrow} & J_s[p^\infty]^{\mathrm{ord}} & \xrightarrow{\twoheadrightarrow} & \alpha(J_s[p^\infty]^{\mathrm{ord}}) \\
\cap\downarrow & & \cap\downarrow & & \cap\downarrow \\
\widehat{A}_s^{\mathrm{ord}} & \xrightarrow{\hookrightarrow} & \widehat{J}_s^{\mathrm{ord}} & \xrightarrow{\twoheadrightarrow} & \alpha(\widehat{J}_s^{\mathrm{ord}}) \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{A}_s^{\mathrm{ord}}/A_s[p^\infty]^{\mathrm{ord}} & \xrightarrow{\hookrightarrow} & \widehat{J}_s^{\mathrm{ord}}/J_s[p^\infty]^{\mathrm{ord}} & \xrightarrow{\twoheadrightarrow} & \alpha(\widehat{J}_s^{\mathrm{ord}})/\alpha(J_s[p^\infty]^{\mathrm{ord}}).
\end{array}$$

The first two terms of the bottom row are sheaves of $\mathbb{Q}_p$-vector spaces, so is the last term. Thus we conclude $\alpha(J_s[p^\infty]^{\mathrm{ord}}) = \alpha(\widehat{J}_s^{\mathrm{ord}})[p^\infty]$. Since $\widehat{A}_s = \widehat{A}_s^{\circ}$, $\widehat{A}_s[p^\infty]^{\mathrm{ord}}$ is a direct summand of the Barsotti–Tate group $J_s[p^\infty]^{\mathrm{ord}}$. Therefore $\alpha(J_s[p^\infty]^{\mathrm{ord}})$ is a Barsotti–Tate group as desired.

Alternatively, we can identify $\alpha_s(\widehat{J}_s^{\mathrm{ord}})[p^\infty]$ with the Barsotti–Tate p-divisible group of the abelian variety quotient J_s/A_s°. □

The condition (A) is a mild condition. Here are sufficient conditions for (α, A_s, B_s) to satisfy (A) given in [H16, Proposition 5.2]:

Proposition 7.3. *Let* $\mathrm{Spec}(\mathbb{T})$ *be a connected component of* $\mathrm{Spec}(\mathbf{h})$ *and* $\mathrm{Spec}(\mathbb{I})$ *be a primitive irreducible component of* $\mathrm{Spec}(\mathbb{T})$. *Then the condition* (A) *holds for the following choices of* (α, A_s, B_s):

(1) *Suppose that an eigen cusp form* $f = f_P$ *new at each prime* $l|N$ *belongs to* $\mathrm{Spec}(\mathbb{T})$ *and that* $\mathbb{T} = \mathbb{I}$ *is regular. Writing the level of* f_P *as* Np^r, *the algebra homomorphism* $\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$ *given by* $f|T(l) = \lambda(T(l))f$ *gives rise to a height* 1 *prime ideal* $P = \mathrm{Ker}(\lambda)$, *which is principal generated by* $a \in \mathbb{T}$. *This* a *has its image* $a_s \in \mathbb{T}_s = \mathbb{T} \otimes_\Lambda \Lambda_s$ *for* $\Lambda_s = \Lambda/(\gamma^{p^{s-\epsilon}} - 1)$. *Write* $\mathbf{h}_s = \mathbf{h} \otimes_\Lambda \Lambda_s = \mathbb{T}_s \oplus 1_s\mathbf{h}_s$ *as an algebra direct sum for an idempotent* 1_s. *Then, the element* $\alpha_s = a_s \oplus 1_s \in \mathbf{h}_s$ *for the identity* 1_s *of* X_s *satisfies* (A). *In this case,* $\alpha = \varprojlim_s \alpha_s$.
(2) *More generally than* (1), *we pick a general connected component* $\mathrm{Spec}(\mathbb{T})$ *of* $\mathrm{Spec}(\mathbf{h})$. *Pick a (classical) Hecke eigenform* $f = f_P$ *(of weight 2) for* $P \in \mathrm{Spec}(\mathbb{T})$. *Assume that* $\mathbf{h}_s$ *(for every* $s \geq r$*) is reduced and* $P = (a)$ *for* $a \in \mathbb{T}$, *and write* a_s *for the image of* a *in* $\mathbf{h}_s$. *Then decomposing* $\mathbf{h}_s = \mathbb{T}_s \oplus 1_s\mathbf{h}_s$, $\alpha_s = a_s \oplus 1_s$ *satisfies* (A).
(3) *Fix* $r > 0$. *Then* α *for a factor* $\alpha|(\gamma^{p^{r-\epsilon}} - 1)$ *in* Λ, *satisfies* (A) *for* $A_s = J_s[\alpha]^{\circ}$ *(the identity connected component).*

Remark 7.4. (i) Under (1), all arithmetic points P of weight 2 in $\mathrm{Spec}(\mathbb{I})$ satisfies (A).

(ii) For a given weight 2 Hecke eigenform f, for density 1 primes $\mathfrak{p}$ of $\mathbb{Q}(f)$, f is ordinary at $\mathfrak{p}$ (i.e., $a(p,f) \not\equiv 0 \mod \mathfrak{p}$; see [H13, §7]). Except for finitely many primes $\mathfrak{p}$ as above, f belongs to a connected component $\mathbb{T}$ which is regular (e.g., [F02, §3.1] and [H16, Theorem 5.3]); so, (1) is satisfied for such $\mathbb{T}$.

8. Mordell–Weil groups of modular abelian factors

Consider the composite morphism $\varpi_s : A_s \hookrightarrow J_s \twoheadrightarrow B_s$ of fppf abelian sheaves for triples (α_s, A_s, B_s) as in (A), and apply the results in Section 6 to abelian varieties $C_s = A_s$ and $D_s = B_s$. Let $\mathcal{C}_s^{\mathrm{ord}} := (\mathrm{Ker}(\varpi_s) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{ord}}$ be the p-primary ordinary part of $\mathrm{Ker}(\varpi_s)$.

Recall we have written ρ_s for the morphism $J_s \to B_s$. As before, κ is an intermediate extension $K/\kappa/k$ finite over k. Define the error terms by

$$E_1^s(\kappa) := \alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)/\alpha(\widehat{J}_s^{\mathrm{ord}}(\kappa)) \quad \text{and} \quad E_2^s(\kappa) := \mathrm{Coker}(\widehat{J}_s^{\mathrm{ord}}(\kappa) \xrightarrow{\rho_s} \widehat{B}_s^{\mathrm{ord}}(\kappa)) \tag{8.1}$$

for $\rho_s : \widehat{J}_s^{\mathrm{ord}}(\kappa) \to \widehat{B}_s^{\mathrm{ord}}(\kappa)$. Note that $E_1^s(\kappa)(\hookrightarrow H^1_?(\widehat{A}_s^{\mathrm{ord}}) = H^1_?(A_s^{\mathrm{ord}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ and $E_2^s(\kappa) = B_s^{\mathrm{ord}}(\kappa)/\rho_s(\widehat{J}_s^{\mathrm{ord}}(\kappa))(\hookrightarrow H^1_?(\alpha(\widehat{J}_s^{\mathrm{ord}}))[\alpha])$ are p-torsion finite module as long as s is finite.

Lemma 8.1. *We have the following commutative diagram with exact rows and exact columns:*

$$\begin{array}{ccccc}
E_1^s(\kappa) & \xrightarrow{\hookrightarrow} & H^1_?(\widehat{A}_s^{\mathrm{ord}}) & \xrightarrow{\iota_s} & H^1_?(\widehat{J}_s^{\mathrm{ord}}) \\
\text{onto}\uparrow & & \uparrow & & \uparrow \\
\frac{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa))} & \xrightarrow{\hookrightarrow} & H^1_?(\mathcal{C}_s^{\mathrm{ord}}) & \xrightarrow{\twoheadrightarrow} & H^1_?(\alpha(\widehat{J}_s^{\mathrm{ord}}))[\alpha] \\
\overline{\alpha}_s\uparrow & & b_s\uparrow & & \uparrow\cup \\
\frac{\widehat{J}_s^{\mathrm{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)} & \underset{\hookrightarrow}{\xrightarrow{\rho_s}} & \widehat{B}_r^{\mathrm{ord}}(\kappa) & \underset{\twoheadrightarrow}{\longrightarrow} & E_2^s(\kappa).
\end{array} \tag{8.2}$$

Each term of the bottom two rows is a profinite module if either k is local or S is a finite set.

The last assertion follows as $\mathcal{C}_s$ is finite and $\widehat{B}_r^{\mathrm{ord}}(\kappa)$ is profinite. We will define each map in the following proof. The proof is the same in any cohomology theory: $H^1_?$ for ? = sm, fppf, étale and Galois cohomology. Therefore, we prove the lemma for the Galois cohomology dropping ? from the notation. This lemma is valid for the Galois cohomology for infinite S as is clear from the proof below.

Proof. Exactness for the bottom row is from the definition of $E_2^s(\kappa)$, and exactness for the left column is by the definition of $E_1^s(\kappa)$. The middle column is a part of the long exact sequence attached to $0 \to \mathcal{C}_s^{\mathrm{ord}} \to \widehat{A}_s^{\mathrm{ord}} \to \widehat{B}_r^{\mathrm{ord}} \to 0$, where $\widehat{B}_s^{\mathrm{ord}}$ is identified with $\widehat{B}_r^{\mathrm{ord}}$ by Lemma 6.2 applied to $D_s = B_s$. The right column comes from the long exact sequence attached to $0 \to \alpha(\widehat{J}_s^{\mathrm{ord}}) \to \widehat{J}_s^{\mathrm{ord}} \to \widehat{B}_r^{\mathrm{ord}} \to 0$, again $\widehat{B}_s^{\mathrm{ord}}$ is identified with $\widehat{B}_r^{\mathrm{ord}}$. The top row comes from the long exact sequence of $0 \to \widehat{A}_s^{\mathrm{ord}} \to \widehat{J}_s^{\mathrm{ord}} \xrightarrow{\alpha} \alpha(\widehat{J}_s^{\mathrm{ord}}) \to 0$.

As for the middle row, we consider the following commutative diagram (with exact rows in the category of fppf abelian sheaves):

$$\begin{array}{ccccc}
\alpha(\widehat{J}_s^{\mathrm{ord}}) & \xrightarrow{\hookrightarrow} & \widehat{J}_s^{\mathrm{ord}} & \underset{\rho_s}{\xrightarrow{\twoheadrightarrow}} & \widehat{B}_s^{\mathrm{ord}} \\
\cup\uparrow & & \cup\uparrow & & \uparrow\| \\
0 \to \alpha(\widehat{J}_s^{\mathrm{ord}}) \times_{\widehat{J}_s^{\mathrm{ord}}} \widehat{A}_s^{\mathrm{ord}} & \underset{\hookrightarrow}{\longrightarrow} & \widehat{A}_s^{\mathrm{ord}} & \underset{\twoheadrightarrow}{\xrightarrow{\varpi_s}} & \widehat{B}_s^{\mathrm{ord}}.
\end{array} \tag{8.3}$$

Under this circumstance, we have $\alpha(\widehat{J}_s^{\mathrm{ord}}) \cap \widehat{A}_s^{\mathrm{ord}} = \alpha(\widehat{J}_s^{\mathrm{ord}}) \times_{\widehat{J}_s^{\mathrm{ord}}} \widehat{A}_s^{\mathrm{ord}} = \mathrm{Ker}(\varpi_s)$ which is a finite étale p-group scheme over $\mathbb{Q}$. Since $\alpha(\widehat{J}_s^{\mathrm{ord}}) \cap \widehat{A}_s^{\mathrm{ord}}$ is equal to $\alpha(\widehat{J}_s^{\mathrm{ord}})[\alpha]$, we have $\mathcal{C}_s^{\mathrm{ord}} = \alpha(\widehat{J}_s^{\mathrm{ord}})[\alpha]$.

Note that $\alpha^2(\widehat{J}_s^{\mathrm{ord}}) = \alpha(\widehat{J}_s^{\mathrm{ord}})$ as sheaves (as $\alpha : \alpha(\widehat{J}_s^{\mathrm{ord}}) \to \alpha(\widehat{J}_s^{\mathrm{ord}})$ is an isogeny, and hence, $\alpha(\alpha(\widehat{J}_s^{\mathrm{ord}}(K))) = \alpha(\widehat{J}_s^{\mathrm{ord}})(K))$. Thus we have a short exact sequence under ?-topology for ? = fppf, sm and ét:

$$0 \to \mathcal{C}_s^{\mathrm{ord}}(K) \to \alpha(\widehat{J}_s^{\mathrm{ord}})(K) \xrightarrow{\alpha} \alpha(\widehat{J}_s^{\mathrm{ord}})(K) \to 0.$$

Look into the associated long exact sequence

$$0 \to \alpha(\widehat{J}_s^{\mathrm{ord}})(k)/\alpha(\alpha(\widehat{J}_s^{\mathrm{ord}})(k)) \to H^1(\alpha(\widehat{J}_s^{\mathrm{ord}})[\alpha]) \to H^1(\alpha(\widehat{J}_s^{\mathrm{ord}})) \xrightarrow{\alpha} H^1(\alpha^2(\widehat{J}_s^{\mathrm{ord}}))$$

which shows the exactness of the middle row, taking the p-primary parts (and then the ordinary parts). □

In the diagram (8.2), we identify $\widehat{A}_s^{\mathrm{ord}}$ with $\widehat{A}_r^{\mathrm{ord}}$ by $\pi_{s,r}^* : J_r \to J_s$ for the projection $\pi_{s,r} : X_s \to X_r$ (Picard functoriality); so, the projective system $\{\widehat{A}_s^{\mathrm{ord}} = \widehat{A}_r^{\mathrm{ord}}, \pi_s^r\}_s$ (w-twisted Albanese functoriality) gives rise to the nontrivial maps $\pi_s^r : \widehat{A}_s^{\mathrm{ord}} = \widehat{A}_r^{\mathrm{ord}} \to \widehat{A}_r^{\mathrm{ord}}$ given by $x \mapsto U(p^{s-r})(p^{s-r}x)$. If we write $H^1(\widehat{A}_r^{\mathrm{ord}}) = (\mathbb{Q}_p/\mathbb{Z}_p)^m \oplus \Delta_r$ for a finite p-torsion group Δ_r by Lemma 2.2 (assuming that S is finite), we have

$$\varprojlim_{\pi_{s\,*}^r : x \mapsto p^{s-r}U(p^{s-r})(x)} H^1(\widehat{A}_r^{\mathrm{ord}}) \cong \varprojlim_{\pi_{s\,*}^r : x \mapsto p^{s-r}x} ((\mathbb{Q}_p/\mathbb{Z}_p)^m \oplus \Delta_r) = \mathbb{Q}_p^m. \tag{8.4}$$

We quote from [CNF, Corollary 2.7.6] the following fact (which is valid also for infinite S):

Lemma 8.2. *We have* $\varprojlim_s H^1(A_r[p^s]^{\mathrm{ord}}) = H^1(T_pA_r^{\mathrm{ord}})$.

We give a proof here for the sake of completeness.

Proof. More generally, let $\{M_n\}_n$ be a projective system of finite $\mathrm{Gal}(k^S/k)$-modules with surjective transition maps. Let $B(M_n)$ (resp. $Z(M_n)$) be the module of 1-coboundaries (resp. continuous 1-cocycles $G \to M_n$ Let $B(M_n)$ (resp. $Z(M_n)$) be the module of 1-coboundaries (resp. inhomogeneous continuous 1-cocycles) $G := \mathrm{Gal}(k^S/k) \to M_n$. We have the exact sequence $0 \to B(M_n) \to Z(M_n) \to H^1(G, M_n) \to 0$. Plainly for $m > n$, the natural map $B(M_m) \to B(M_n)$ is onto. Thus the above sequences satisfies the Mittag–Leffler condition, and plainly $\varprojlim_n ?(M_n) = ?(\varprojlim_n M_n)$ for $? = B, Z$, we have $\varprojlim_n H^1(k^S/k, M_n) = H^1(k^S/k, \varprojlim_n M_n)$. □

We have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathcal{C}_s^{\mathrm{ord}} & \hookrightarrow & \widehat{A}_s^{\mathrm{ord}} & \twoheadrightarrow & \widehat{B}_s^{\mathrm{ord}} \\ \downarrow & & \downarrow & & \wr\downarrow \\ \mathcal{C}_r^{\mathrm{ord}} & \hookrightarrow & \widehat{A}_r^{\mathrm{ord}} & \twoheadrightarrow & \widehat{B}_r^{\mathrm{ord}}. \end{array}$$

By the snake lemma applied to the above diagram, we get the following exact sequence:

$$0 \to A_r[p^{s-r}]^{\mathrm{ord}} \to \mathcal{C}_s^{\mathrm{ord}} \to \mathcal{C}_r^{\mathrm{ord}} \to 0.$$

Passing to the limit (as continuous H^1 for profinite coefficients is a projective limit of H^1 of finite coefficients; cf., [CNF, 2.7.6]), we have

$$T_pA = \varprojlim_s A_r[p^s]^{\mathrm{ord}} = \varprojlim_s \mathcal{C}_s^{\mathrm{ord}} \text{ and } H^1(T_pA_r^{\mathrm{ord}}) = \varprojlim_s H^1(A_r[p^s]^{\mathrm{ord}}) = \varprojlim_s H^1(\mathcal{C}_s^{\mathrm{ord}}). \tag{8.5}$$

9. Control theorems with an error term

Taking the projective limit of the exact sequence $0 \to \widehat{A}_s^{\mathrm{ord}} \to \widehat{J}_s^{\mathrm{ord}} \xrightarrow{\alpha} \widehat{J}_s^{\mathrm{ord}}$, by the vanishing $\varprojlim_s \widehat{A}_s^{\mathrm{ord}}(\kappa) = 0$ in Proposition 6.4 applied to $C_s = A_s$, we get the injectivity of $\widehat{J}_\infty^{\mathrm{ord}} \xrightarrow{\alpha} \widehat{J}_\infty^{\mathrm{ord}}$.

Since all the terms of the exact sequences: $0 \to \alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa) \to \widehat{J}_s^{\mathrm{ord}}(\kappa) \to \frac{\widehat{J}_s^{\mathrm{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)} \to 0$ are compact p-profinite groups, after taking the limit with respect to π_s^r, we still have an exact sequence

$$0 \to \varprojlim_s \alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa) \to \varprojlim_s \widehat{J}_s^{\mathrm{ord}}(\kappa) \to \varprojlim_s \frac{\widehat{J}_s^{\mathrm{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)} \to 0$$

with $\varprojlim_s \frac{\widehat{J}_s^{\mathrm{ord}}(\kappa)}{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)} \hookrightarrow \widehat{B}_r^{\mathrm{ord}}(\kappa)$. Thus

$$\frac{\widehat{J}_\infty^{\mathrm{ord}}(\kappa)}{\alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa)} := \frac{\varprojlim_s \widehat{J}_s^{\mathrm{ord}}(\kappa)}{\varprojlim_s \alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)} \cong \varprojlim_s \frac{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa))}.$$

Here the last isomorphism follows from the injectivity of α. By the same token, we have

$$\frac{\alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa))} := \frac{\varprojlim_s \alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)}{\varprojlim_s \alpha(\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa))} = \varprojlim_s \frac{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa))}.$$

Writing $E_j^\infty(\kappa) = \varprojlim_s E_j^s(\kappa)$ and passing to projective limit of the diagram (8.2), we get the following commutative diagram with exact rows:

$$(9.1)\qquad \begin{array}{ccccc}
E_1^\infty(\kappa) & \xrightarrow[\text{Lemma 8.1}]{\hookrightarrow} & \varprojlim_{s:x\mapsto p^{s-r}U(p^{s-r})(x)} H^1(\widehat{A}_r^{\mathrm{ord}}) & \xrightarrow{\iota_\infty} & H^1(\varprojlim_s \widehat{J}_s^{\mathrm{ord}}(K)) \\
\text{onto}\uparrow & & \uparrow & & \uparrow \\
\frac{\alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa))} & \xrightarrow{\hookrightarrow} & H^1(T_pA_r(K)^{\mathrm{ord}}) & \xrightarrow{a} & \varprojlim_s H^1(\alpha(\widehat{J}_s^{\mathrm{ord}})(K))[\alpha] \\
\overline{\alpha}_\infty\uparrow & & b\uparrow\cup & & c\uparrow\cup \\
\frac{\widehat{J}_\infty^{\mathrm{ord}}(\kappa)}{\alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa)} & \xrightarrow[\hookrightarrow]{\rho_s} & \widehat{B}_r^{\mathrm{ord}}(\kappa) & \xrightarrow{d} & E_2^\infty(\kappa).
\end{array}$$

The rows are exact since projective limit is left exact. The maps a and d are onto if either S is finite or k is local (as projective limit is exact for profinite modules). By the same token, the right and left columns are also exact. Therefore $E_j^\infty(\kappa)$ $(j = 1, 2)$ is a torsion Λ-module of finite type.

To see, we look into the cohomology exact sequence of the short exact sequence: $\mathcal{C}_s \hookrightarrow \widehat{A}_r^{\mathrm{ord}} \twoheadrightarrow \widehat{B}_r^{\mathrm{ord}}$ with transition maps $p^{s'-s}U(p)^{s'-s}$ for $\{\mathcal{C}_s\}_s$ and $\{\widehat{A}_r^{\mathrm{ord}}\}_s$ and $U(p)^{s'-s}$ for $\{\widehat{B}_r^{\mathrm{ord}}\}_s$. Thus we have the limit sequence

$$0 \to \varprojlim_{s:x\mapsto p^{s-r}U(p^{s-r})(x)} \widehat{A}_r^{\mathrm{ord}}(\kappa)/\mathcal{C}_s(\kappa) \to \widehat{B}_r^{\mathrm{ord}}(\kappa) \xrightarrow{b} \varprojlim_s H^1(\mathcal{C}_s) = H^1(T_pA_r^{\mathrm{ord}}).$$

This sequence is exact as all the terms are profinite compact modules at each step. Since

$$\varprojlim_{s:x\mapsto p^{s-r}U(p^{s-r})(x)} \widehat{A}_r^{\mathrm{ord}}(\kappa)/\mathcal{C}_s(\kappa) = 0,$$

the map b is injective.

Passing to the limit of exact sequences of profinite modules: $\mathcal{C}_s(\kappa) \to \widehat{A}_r^{\mathrm{ord}}(\kappa) \xrightarrow{\varpi_s} \widehat{B}_r^{\mathrm{ord}}(\kappa) \twoheadrightarrow \mathrm{Coker}(\varpi_s)$, we get the limit exact sequence $0 \to \widehat{B}_r^{\mathrm{ord}}(\kappa) \cong \varprojlim_s \mathrm{Coker}(\varpi_s)$. By the left exactness of projective limit, the sequence

$$0 \to \varprojlim_s \mathrm{Coker}(\varpi_s) \to H^1(T_pA_r^{\mathrm{ord}}) \to \varprojlim_s H^1(\widehat{A}_r^{\mathrm{ord}})$$

is exact. Therefore the middle column is exact; so,

$$(9.2)\qquad \overline{\alpha}_\infty \text{ is injective.}$$

Since $\widehat{J}_\infty(\kappa)^{\mathrm{ord}}[\alpha] = \widehat{A}_\infty^{\mathrm{ord}}(\kappa) = 0$, $\alpha : \alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa) \to \alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa)$ is injective.

This shows

Lemma 9.1. *Let κ be a field extension of $\mathbb{Q}$ or $\mathbb{Q}_l$ for a prime l, but we assume finiteness condition* (2.2) *for the extension κ/k. We allow an infinite set S of places of k when k is finite extension of $\mathbb{Q}$. Let α be as in* (A). *Then we have the following exact sequences (of p-profinite Λ-modules) up to Λ-torsion error:*

$$0 \to \widehat{J}_\infty^{\mathrm{ord}}(\kappa) \xrightarrow{\alpha} \alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa) \to E_1^\infty(\kappa)^{\mathrm{ord}} \to 0$$

and

$$0 \to \alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa) \to \widehat{J}_\infty^{\mathrm{ord}}(\kappa) \xrightarrow{\rho_\infty} \widehat{B}_r^{\mathrm{ord}}(\kappa) \to E_2^\infty(\kappa) \to 0.$$

Here $E_j^\infty(\kappa)$ is a Λ-torsion module of finite type. In particular, taking $\alpha = \gamma - 1$, we conclude that the compact module $\widehat{J}_\infty(\kappa)$ is a Λ–module of finite type.

The statement of this lemma is independent of the set S (though in the proof, we used Galois cohomology groups for finite S if k is global); therefore, the lemma is valid also for an infinite set S of places of k (as long as S contains all p-adic and archimedean places and places over N).

The left column of (9.1) is made up of compact modules for which projective limit is an exact functor; so, left column is exact; in particular

$$\varprojlim_s \frac{\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(\widehat{J}_s^{\mathrm{ord}})(\kappa))} \to E_1^\infty(\kappa) := \varprojlim_s E_1^s(\kappa)$$

is onto.

Take the maximal Λ-torsion module X inside $\widehat{J}_\infty^{\mathrm{ord}}(\kappa)$. Since X is unique, it is an $\mathbf{h}$-module. The module $\widehat{J}_\infty^{\mathrm{ord}}(\kappa)$ is pseudo-isomorphic to $X \oplus \Lambda^R$ for a positive integer R. Since α is injective on $\widehat{J}_\infty^{\mathrm{ord}}(\kappa)$, for the α-localization $\mathbf{h}_{(\alpha)}$, we have $X_\alpha = X \otimes_{\mathbf{h}} \mathbf{h}_{(\alpha)} = 0$. Thus $\widehat{J}_\infty^{\mathrm{ord}}(\kappa) \otimes_{\mathbf{h}} \mathbf{h}_{(\alpha)}$ is Λ_{P_α}-free, where $P_\alpha = (\alpha) \cap \Lambda$. Thus $\frac{\alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa)}{\alpha(\alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa))} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\frac{\widehat{J}_\infty^{\mathrm{ord}}(\kappa)}{\alpha(\widehat{J}_\infty^{\mathrm{ord}})(\kappa)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ have equal $\mathbb{Q}_p$-dimension. Therefore, by the injectivity of $\overline{\alpha}_\infty$ (9.2), $E_1^\infty(\kappa)$ is p-torsion. However by (8.4), this torsion module is embedded in a $\mathbb{Q}_p$-vector space by the top sequence of (9.1), we have $E_1^\infty(\kappa) = 0$. This shows

Theorem 9.2. *Let α be as in* (A) *and k be a finite field extension of either $\mathbb{Q}$ or $\mathbb{Q}_l$ for a prime l. Assume* (2.2) *for the extension κ/k. Then we have the following exact sequence (of p-profinite Λ-modules):*

$$0 \to \widehat{J}_\infty^{\mathrm{ord}}(\kappa) \xrightarrow{\alpha} \widehat{J}_\infty^{\mathrm{ord}}(\kappa) \xrightarrow{\rho_\infty} \widehat{B}_r^{\mathrm{ord}}(\kappa) \to E_2^\infty(\kappa) \to 0.$$

In particular, taking $\alpha = \gamma - 1$, we conclude that the Λ-module $\widehat{J}_\infty(\kappa)$ is a Λ–module of finite type and that $\widehat{J}_\infty(\kappa)$ does not have any pseudo-null Λ-submodule non null (i.e., $\widehat{J}_\infty(\kappa)$ has Λ-homological dimension ≤ 1).

By this theorem (applied to $\alpha = \gamma^{p^s} - 1$ for $s = 1, 2, \dots$), the localization $\widehat{J}_\infty(\kappa)_P$ at an arithmetic prime P is Λ_P-free of finite rank, which also follows from [N06, Proposition 12.7.13.4] as $\widehat{J}_\infty(\kappa)$ can be realized inside Nekovář's Selmer group by the embedding of Lemma 2.1.

10. Control theorem for a number field

The following theorem is the final result of this paper for a number field k.

Theorem 10.1. *Let the notation be as in the introduction. Suppose that k is a finite extension of $\mathbb{Q}$. Let $\mathcal{A}_{\mathbb{T}}$ be the set of all principal arithmetic points of* $\mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$ *of weight* 2 *and put $\Omega_{\mathbb{T}} := \{P \in \mathcal{A}_{\mathbb{T}} | A_P \text{ has good reduction over } \mathbb{Z}_p[\mu_{p^\infty}]\}$. Suppose that we have a single point $P_0 \in \Omega_{\mathbb{T}}$ with finite* $\mathrm{Sel}_k(A_{P_0})^{\mathrm{ord}}$*, and write* $\mathrm{Spec}(\mathbb{I})$ *for the unique irreducible component on which P_0 lies. Let k be a finite field extension of either $\mathbb{Q}$ or $\mathbb{Q}_l$ for a prime l. Then, for almost all $P \in \Omega_{\mathbb{T}} \cap \mathrm{Spec}(\mathbb{I})$, we have the following exact sequence (of p-profinite Λ-modules):*

$$0 \to \widehat{J}_{\infty,\mathbb{T}}^{\mathrm{ord}}(k) \xrightarrow{\alpha} \widehat{J}_{\infty,\mathbb{T}}^{\mathrm{ord}}(k) \xrightarrow{\rho_\infty} \widehat{B}_P^{\mathrm{ord}}(k) \to E_2^\infty(k) \to 0$$

with finite error term $|E_2^\infty(k)| < \infty$.

Since $\mathbb{T}$ is étale at P_0 over Λ, only one irreducible component of $\mathrm{Spec}(\mathbb{T})$ contains P_0 (e.g. [HMI, Proposition 3.78]).

Since the root number of $L(s, A_P)$ is not equal to -1 for most points (as $\{X_r\}_r$ is the standard tower), we expect that $|\mathrm{Sel}_k(A_P)^{\mathrm{ord}}| < \infty$ for most arithmetic P; so, the assumption of the theorem is a reasonable one.

Proof. The Selmer group $\mathrm{Sel}_k(A_P)^{\mathrm{ord}}$ is the one defined in [H16, §8]. By [N06, 12.7.13.4] or [H16, Theorem A], the finiteness $|\mathrm{Sel}_k(A_{P_0})^{\mathrm{ord}}| < \infty$ for a single point $P_0 \in \Omega_{\mathbb{T}}$ implies that $\mathrm{Sel}_k(A_P)^{\mathrm{ord}}$ is finite for almost all $P \in \Omega_{\mathbb{T}} \cap \mathrm{Spec}(\mathbb{I})$. Though in [H16, Theorem A], it is assumed that $\mathbb{T}$ is regular to guarantee that all arithmetic points are principal, what we need to get the result is the principality of P_0 and P in $\mathbb{T}$; so, this holds true for $P \in \Omega_{\mathbb{T}} \cap \mathrm{Spec}(\mathbb{I})$. By the well known exact sequence

$$0 \to \widehat{B}^{\mathrm{ord}}_P(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_k(A_P)^{\mathrm{ord}} \to Ш_k(A_P)^{\mathrm{ord}} \to 0,$$

the finiteness of $\mathrm{Sel}_k(A_P)^{\mathrm{ord}}$ implies finiteness of $\widehat{B}^{\mathrm{ord}}_P(k)$; so, $E_2(k)$ is finite as well. □

11. Local error term

Now let k be an l-adic field. As before, we write $H^q(M)$ for $H^q(k, M)$. For any abelian variety $X_{/k}$, we have an exact sequence $\widehat{X}(k) \hookrightarrow H^1(T_pX) \twoheadrightarrow \varprojlim_n H^1(\widehat{X})[p^n]$ by Lemma 2.1. Similarly, by Corollary 7.2, Lemma 2.1 tells us that $\alpha(\widehat{J}^{\mathrm{ord}}_s(k)) \hookrightarrow H^1(T_p\alpha(\widehat{J}^{\mathrm{ord}}_s)) \twoheadrightarrow T_pH^1(\alpha(\widehat{J}^{\mathrm{ord}}_s)) := \varprojlim_n H^1(\alpha(\widehat{J}^{\mathrm{ord}}_s))[p^n]$ is exact. Thus we have the following commutative diagram in which the first two columns and the first three rows are exact by Lemma 8.2 and left exactness of the formation of projective limits combined (the surjectivity of the three horizontal arrows c_j $(j = 1, 2, 3)$ are valid if S is finite or k is local):

$$\begin{array}{ccccc}
\alpha(\widehat{J}^{\mathrm{ord}}_s)(k) & \xrightarrow{\hookrightarrow} & H^1(T_p\alpha(\widehat{J}^{\mathrm{ord}}_s)) & \xrightarrow[\twoheadrightarrow]{c_1} & T_pH^1(\alpha(\widehat{J}^{\mathrm{ord}}_s)) \\
\cap\downarrow i & & a\downarrow & & \downarrow b \\
\widehat{J}^{\mathrm{ord}}_s(k) & \xrightarrow[f]{\hookrightarrow} & H^1(T_pJ^{\mathrm{ord}}_s) & \xrightarrow[\twoheadrightarrow]{c_2} & T_pH^1(\widehat{J}^{\mathrm{ord}}_s) \\
\rho_s\downarrow & & j\downarrow & & \downarrow h \\
\widehat{B}^{\mathrm{ord}}_r(k) & \xrightarrow[\beta]{\hookrightarrow} & H^1(T_pB^{\mathrm{ord}}_r) & \xrightarrow[\twoheadrightarrow]{c_3} & T_pH^1(\widehat{B}^{\mathrm{ord}}_r) \\
\text{onto}\downarrow \pi & & \varpi_s\downarrow & & \downarrow g \\
E^s_2(k) & \xrightarrow{e_s} & H^2(T_p\alpha(\widehat{J}^{\mathrm{ord}}_s)) & \longrightarrow & T_pH^2(\alpha(\widehat{J}^{\mathrm{ord}}_s)).
\end{array} \tag{11.1}$$

Assuming that S is finite, the right column is made of $\mathbb{Z}_p$-free modules, and hence, the rows are split exact sequences.

To see the existence of the map e_s, we suppose that $x = \rho_s(y) \in \mathrm{Im}(\rho_s)$. Then we have

$$\varpi_s(\beta(x)) = \varpi_s(\beta(\rho_s(y))) = \varpi_s(j(f(y))) = 0.$$

If $b \equiv b' \mod \mathrm{Im}(\rho_s)$ for $b, b' \in \widehat{B}^{\mathrm{ord}}_r(k)$, we have $\varpi_s(\beta(b)) = \varpi_s(\beta(b'))$. In other words, $\pi(b) \mapsto \varpi(\beta(b))$ is a well-defined homomorphism from $E^s_2(k) \cong \widehat{B}^{\mathrm{ord}}_r(k)/\mathrm{Im}(\rho_s)$ into $\mathrm{Im}(\varpi_s) \cong \mathrm{Coker}(j) \subset H^2(T_p\alpha(\widehat{J}^{\mathrm{ord}}_s))$, which we have written as e_s.

We have the following fact (cf. [H15, Corollary 4.4]).

Lemma 11.1. *We have $H^0(T_pB^{\mathrm{ord}}_r) = H^0(T\mathcal{G}) = 0$, where $T\mathcal{G} := \mathrm{Hom}_\Lambda(\Lambda^\vee, \mathcal{G}) \cong \varprojlim_s T_pJ^{\mathrm{ord}}_s$.*

Proof. We only need to prove this for a finite field extension k of $\mathbb{Q}_l$ (as this implies the result for a number field) and T_pB_r (as we can take $B_r := J_r$, which implies the result for $T\mathcal{G}$). Write $B = B_r$. By replacing k be a finite field extension, we may assume that B has either good reduction or split multiplicative reduction over the valuation ring O of k with residue field $\mathbb{F}$. If B has good reduction over O and $l \neq p$, T_pB^{ord} is unramified at l. All the eigenvalues of the action of the l-Frobenius element ϕ are a Weil l-number of positive weight; so, we conclude

$$H^0(T_pB) \subset \mathrm{Ker}(\phi - 1 : T_pB \to T_pB) = 0,$$

and the assertion follows.

Modifying B by an isogeny does not affect the outcome; so, by doing this, we may assume that $\mathrm{End}(B_{/\mathbb{Q}})$ contains the integer ring O_B of the Hecke field. Suppose that $p = l$, and take a prime factor $\mathfrak{p}|p$ in O_B such that $T_pB^{\mathrm{ord}} = T_{\mathfrak{p}}B := \varprojlim_n B[\mathfrak{p}^n](\overline{\mathbb{Q}})$. Then $B[\mathfrak{p}^\infty]^{\mathrm{ord}}$ extends to an ordinary Barsotti–Tate group. If B does not have complex multiplication, by [Z14], the connected-étale exact sequence

$$0 \to B[\mathfrak{p}]^{\circ,\mathrm{ord}} \to B[\mathfrak{p}^\infty]^{\mathrm{ord}} \to B[\mathfrak{p}^\infty]^{\text{ét}} \to 0$$

is non-split as a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/k)$–module; so, $H^0(T_{\mathfrak{p}}B^{\mathrm{ord}}) = 0$ again. If B has complex multiplication, by the Cartier duality, we have a Galois equivariant non-degenerate pairing

$$(T_pB[\mathfrak{p}^\infty]^{\text{ét}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \times (T_pB[\mathfrak{p}]^{\circ,\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \to \mathbb{Q}_p(1).$$

On $T_pB[\mathfrak{p}^\infty]^{\text{ét}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, again the eigenvalues of the action of the p-Frobenius element ϕ are Weil p-numbers of positive weight. This shows $H^0(T_pB[\mathfrak{p}^\infty]^{\text{ét}}) = 0$. By duality, $H^0(T_pB[\mathfrak{p}^\infty]^{\circ,\mathrm{ord}}) = 0$. Then from the exact sequence

$$0 \to T_pB[\mathfrak{p}^\infty]^{\circ,\mathrm{ord}} \to T_pB^{\mathrm{ord}} \to T_pB[\mathfrak{p}^\infty]^{\text{ét}} \to 0,$$

we conclude $H^0(T_pB^{\mathrm{ord}}) = 0$.

If B is split multiplicative over O, this fact is a well known result of Mumford–Tate [Mu72]. □

By the above lemma, the map a in (11.1) is injective.

Lemma 11.2. *Let k be either a number field or a finite extension of $\mathbb{Q}_l$ for a prime l. Then the map b in the diagram* (11.1) *is injective, and if k is local with $l \neq p$, we have* $\mathrm{Im}(b) = \mathrm{Ker}(h) = 0$ *in* (11.1) *(so the right column is exact).*

Proof. Applying the snake lemma to the first two rows of (11.1), we find that b is injective.

Suppose that k is local. For an abelian variety X over k with $X^t := \mathrm{Pic}^0_{X/k}$, $X^t(k)$ is isomorphic to $\mathbb{Z}_l^m$ times a finite group; so, if $l \neq p$, $\widehat{X}^t(k)$ is finite p-group. By [ADT, I.3.4], $H^1(k, X) \cong X^t(k)^\vee$; so, $H^1(k, \widehat{X})$ is a finite group. Therefore $H^1(k, \widehat{J}_s^{\mathrm{ord}})$ and $H^1(k, \widehat{B}_r^{\mathrm{ord}})$ are finite groups, and $T_pH^1(k, \widehat{J}_s^{\mathrm{ord}}) = T_pH^1(k, \widehat{B}_r^{\mathrm{ord}}) = 0$. Since b is injective, $T_pH^1(k, \alpha(\widehat{J}_s^{\mathrm{ord}})) = 0$; so, $\mathrm{Ker}(h) = \mathrm{Im}(b) = 0$. □

We note the following fact: If k is local non-archimedean, for an abelian variety A over k,

$$H^2(k, \widehat{A}) = H^2(k, A) = 0 \ \text{ for any abelian variety } A \text{ over } k. \tag{11.2}$$

This follows from [ADT, Theorem I.3.2], since $H^2(k, \widehat{A}) = H^2(k, A) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Proposition 11.3. *If k be a finite extension of $\mathbb{Q}_l$ with $l \neq p$, then $E_2^s(k) = 0$.*

Proof. Since the left column of (11.1) by Lemma 11.2 if $l \neq p$, applying the snake lemma to the middle two exact rows of (11.1), we find an exact sequence

$$0 \to E_2^s(k) \xrightarrow{e_s} \mathrm{Im}(\varpi_s) \to \mathrm{Coker}(h) \to 0. \tag{11.3}$$

This implies $E_2^s(k) \hookrightarrow \mathrm{Im}(\varpi_s)$.

Let $X_{/k}$ be a p-divisible Barsotti–Tate group. We have $H^2(k, T_pX) = \varprojlim_n H^2(k, X[p^n])$ (e.g., [CNF, 2.7.6]). By Tate duality (e.g., [MFG, Theorem 4.43]), we have $H^2(k, X[p^n]) \cong X^t[p^n](k)^\vee$ for the Cartier dual $X^t := \mathrm{Hom}(T_pX, \mu_{p^\infty})$ of X. Thus we have

$$H^2(k, T_pX) = \varprojlim_n (X^t[p^n](k)^\vee) \cong (\varinjlim_n H^0(k, X^t[p^n]))^\vee,$$

since we have a canonical pairing $X[p^n] \times X^t[p^n] \to \mu_{p^n}$ (i.e., $X^t[p^n](k)^\vee \cong X[p^n](-1)(k)$).

Apply this to the complement X of $\widehat{A}_s[p^\infty]^{\mathrm{ord}}$ in $J_s[p^\infty]^{\mathrm{ord}}$; so, $X + A_s[p^\infty]^{\mathrm{ord}} = J_s[p^\infty]^{\mathrm{ord}}$ with finite $X \cap A_s[p^\infty]^{\mathrm{ord}}$. Requiring X to be stable under $\mathbf{h}_s$, for $\mathbf{h}_s(\mathbb{Q}_p) = \mathbf{h}_s \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, X is uniquely determined as $\mathbf{h}_s(\mathbb{Q}_p) = (\mathbf{h}_s(\mathbb{Q}_p)/\alpha_s\mathbf{h}_s(\mathbb{Q}_p)) \oplus 1_s\mathbf{h}_s(\mathbb{Q}_p)$ for an idempotent 1_s (so, $X = 1_sJ_s[p^\infty]^{\mathrm{ord}}$). By local Tate duality, we get $H^2(k, T_pX) \cong H^0(k, X[p^\infty]^t)^\vee$ and conclude

$$H^2(k, T_pX) \cong \varinjlim_n H^0(k, \mathrm{Hom}(X[p^n](\overline{k}), \mu_{p^n}(\overline{k}))) = \varinjlim_n X[p^n](-1)(k) = X[p^\infty](-1)(k).$$

Thus we conclude the injectivity:

$$H^2(k, T_pX) \cong X[p^\infty](-1)(k) \xrightarrow{\hookrightarrow} J_s[p^\infty]^{\mathrm{ord}}(-1)(k) \cong H^2(k, T_pJ_s)^{\mathrm{ord}},$$

which is injective as $X \subset J_s[p^\infty]^{\mathrm{ord}}$. By definition, we have $X + A_s[p^\infty]^{\mathrm{ord}} = J_s[p^\infty]^{\mathrm{ord}}$. By the assumption (A) and the definition of X, $X = \alpha_s(J_s[p^\infty]^{\mathrm{ord}})$. Therefore we get an injection:

$$H^2(k, T_p\alpha(\widehat{J}_s^{\mathrm{ord}})) \cong H^2(k, T_p\alpha(J_s[p^\infty]^{\mathrm{ord}})) \cong \alpha(J_s[p^\infty]^{\mathrm{ord}})(-1)(k) \xhookrightarrow{a_2} J_s[p^\infty]^{\mathrm{ord}}(-1)(k) \cong H^2(k, T_pJ_s)^{\mathrm{ord}}.$$

We have an exact sequence

$$H^1(k, T_p\alpha(\widehat{J}_s^{\mathrm{ord}})) \xrightarrow{\varpi_s} H^2(k, T_pA_r)^{\mathrm{ord}} \xrightarrow{a_2} H^2(k, T_pJ_s)^{\mathrm{ord}}.$$

Since a_2 is injective, we find $\mathrm{Im}(\varpi_s) = 0$; so, $E_2^s(k) = 0$ if k is l-adic with $l \neq p$. □

Here are some remarks what happens when $l = p$ for the local error terms. For simplicity, we assume that $k = \mathbb{Q}_p$; so, $W_s = \mathbb{Z}_p[\mu_{p^s}]$. For $l \neq p$, the proof of the above proposition is an argument purely of characteristic 0. In [H16, §17], we studied the error term of the control of inductive limit $J_\infty^{\mathrm{ord}}(\mathbb{Q}_p) := \varinjlim_s \widehat{J}_s^{\mathrm{ord}}(\mathbb{Q}_p)$ using a result of P. Schneider [Sc83] and [Sc87] on universal norm for abelian varieties over ramified $\mathbb{Z}_p$-extension. It works well for the inductive limit $J_\infty^{\mathrm{ord}}(\mathbb{Q}_p)$ but perhaps not for the projective limit $\widehat{J}_\infty^{\mathrm{ord}}(\mathbb{Q}_p)$ for the following reason.

This involves study of integral models of the abelian variety (in particular, its formal Lie group over W_∞). Let I_r (resp. $X_{r,0}$) be the Igusa tower of level p^r over $X_0 := X_1(N) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ containing the zero cusp (resp. the infinity cusp). Then for $P \in \Omega_{\mathbb{T}}$, if the conductor of B_P is divisible by p^r with $r > 0$, $B_P \times_{W_r} \mathbb{F}_p$ is the quotient of $\mathrm{Pic}^0_{I_r/\mathbb{F}_p} \times \mathrm{Pic}^0_{X_{r,0}/\mathbb{F}_p}$ (cf. [AME, Chapter 14] or [H14, §6]). On $\mathrm{Pic}^0_{I_r/\mathbb{F}_p} \times \mathrm{Pic}^0_{X_{r,0}/\mathbb{F}_p}$, $U(p)$ and $U^*(p)$ acts in a matrix form with respect to the two factors $\mathrm{Pic}^0_{I_r/\mathbb{F}_p}$ and $\mathrm{Pic}^0_{X_{r,0}/\mathbb{F}_p}$ in this order

$$U(p) = \begin{pmatrix} F & * \\ 0 & V\langle p^{(p)}\rangle \end{pmatrix} \text{ and } U^*(p) = \begin{pmatrix} V\langle p^{(p)}\rangle & 0 \\ * & F \end{pmatrix}, \tag{11.4}$$

where $\langle p^{(p)}\rangle$ is the diamond operator of the class of p modulo N. See [MW84, §3.3] or [H14, (6-1)] for this formula. Since the shoulder term $*$ of the above matrix form of $U(p)$ vanishes once restricted to B_P if $r > 0$, from (11.4), $\widehat{B}_P^{\mathrm{ord}}(\mathbb{F})$ must be the quotient of $\mathrm{Pic}^0_{I_r/\mathbb{F}_p}$, and the ordinary part of the formal Lie group $\widehat{B}^{\circ,\mathrm{ord}}_{P/\mathbb{F}_p}$ of $\widehat{B}_P$ has to be the quotient of $\mathrm{Pic}^0_{X_{r,0}/\mathbb{F}_p}$. Similarly, $\widehat{B}_P^{\text{co-ord}}(\mathbb{F})$ must be the quotient of $\mathrm{Pic}^0_{X_{r,0}/\mathbb{F}_p}$, and the co-ordinary part of the formal Lie group $\widehat{B}^{\circ,\text{co-ord}}_{P/\mathbb{F}_p}$ of $\widehat{B}_P$ has to be the quotient of $\mathrm{Pic}^0_{I_r/\mathbb{F}_p}$.

Write B_s for the quotient of J_s corresponding to B_P. We consider the exact sequence defining $E_2^s(\mathbb{Q}_p)$:

$$0 \to \alpha(\widehat{J}^{\mathrm{ord}}_{s,\mathbb{T}})(\mathbb{Q}_p) \to \widehat{J}^{\mathrm{ord}}_{s,\mathbb{T}}(\mathbb{Q}_p) \xrightarrow{\rho_s} \widehat{B}_s^{\mathrm{ord}}(\mathbb{Q}_p) \to E_2^s(\mathbb{Q}_p) \to 0,$$

which is equivalent to, by the involution w_s over characteristic 0 field, the following exact sequence

$$0 \to \alpha^*(\widehat{J}^{\text{co-ord}}_{s,\mathbb{T}})(\mathbb{Q}_p) \to \widehat{J}^{\text{co-ord}}_{s,\mathbb{T}}(\mathbb{Q}_p) \xrightarrow{\rho_s^*} {}^t\widehat{A}_s^{\text{co-ord}}(\mathbb{Q}_p) \to \mathcal{E}_2^s(\mathbb{Q}_p) \to 0.$$

Thus we study the second exact sequence of the co-ordinary parts. Here we have used the self duality of J_s, ${}^tA_s$ is the dual abelian variety of A_s and α^* is the image of α under the Rosati involution.

Consider the complex of Néron models over W_s:

$$0 \to \alpha^*(J_s) \to J_s \to {}^tA_s \to 0$$

and its formal completion along the identity

$$0 \to \alpha^*(J_s^\circ) \to J_s^\circ \to {}^tA_s^\circ \to 0.$$

Here X° is the formal group of an abelian variety $X_{/W_s}$. These sequence might not be exact as $W_s/\mathbb{Z}_p$ is highly ramified at p (see [NMD, §7.5]). But just to go forward, we assume the sequence of the co-ordinary parts of the formal Lie groups are exact (and still we find some difficulties).

As explained in [H16, (17.3)], taking the $\mathbb{T}^*$-component (the image of $\mathbb{T}$ under the Rosati involution), the complex

$$0 \to \alpha^*(\widehat{J}^\circ_{s,\mathbb{T}^*}) \to \widehat{J}^\circ_{s,\mathbb{T}^*} \to {}^t\widehat{A}^\circ_{s,\mathbb{T}^*} \to 0 \tag{11.5}$$

is, by our assumption, an exact sequence of formal Lie groups over W_s; so, the top complex of the following commutative diagram is a short exact sequence:

$$\begin{array}{ccccc}\alpha^*(\widehat{J}^\circ_{s,\mathbb{T}^*})(W_s) & \xrightarrow{\hookrightarrow} & \widehat{J}^\circ_{s,\mathbb{T}^*}(W_s) & \xrightarrow{\twoheadrightarrow} & {}^t\widehat{A}^\circ_{s,\mathbb{T}^*}(W_s) \\ {\scriptstyle N_{\alpha^*(J_s)}}\downarrow & & {\scriptstyle N_{J_s}}\downarrow & & {\scriptstyle N_s}\downarrow \\ \alpha^*(\widehat{J}^\circ_{s,\mathbb{T}^*})(W_s)^{\mathrm{Gal}(k_s/k_r)} & \xrightarrow{\hookrightarrow} & \widehat{J}^\circ_{s,\mathbb{T}^*}(W_s)^{\mathrm{Gal}(k_s/k_r)} & \xrightarrow{\rho_s^*} & {}^t\widehat{A}^\circ_{r,\mathbb{T}^*}(W_r),\end{array}$$

where $N_{X,s}$ is the norm map relative to k_s/k_r of an abelian variety X defined over k_r. By Schneider [Sc87], N_s is almost onto with the index of the image bounded independent of s. However, we do not know yet ρ_s^* is surjective up to finite bounded error for the following reason:

Though ${}^t\widehat{A}_s^{\text{co-ord}} \cong {}^t\widehat{A}_r^{\text{co-ord}}$ *because* $\widehat{A}_r^{\text{ord}} \cong \widehat{A}_s^{\text{ord}}$ *as seen in* [H16], *the projection map*

$${}^t\widehat{A}_s^{\circ,\text{ co-ord}}(W_s) \to {}^t\widehat{A}_r^{\circ,\text{ co-ord}}(W_s)$$

is not an isomorphism. After reducing modulo p, *as already remarked, the formal Lie group of* ${}^t\widehat{A}_s^{\text{co-ord}}$ *is in the identity connected component of* $\mathrm{Pic}^0_{X_{0,s}}$. *Note that* $I_s = X_{s,0}^{(p^s)}$ *(the* p^s*-power Frobenius twist); so, the projection* $I_s \to I_r$ *is given by* $F^{s-r} \circ \pi$ *for the projection* $\pi : X_{s,0} \to X_{r,0}$ *(*[AME, Theorem 13.11.4 (1)] *or* [H14, §6]*) which is purely inseparable. This shows that* $\pi_* : {}^t\widehat{A}_s^{\text{co-ord}} \to {}^t\widehat{A}_r^{\text{co-ord}}$ *is not an isomorphism.* Thus we have two problems for proving bounded-ness of $\mathcal{E}_2^s(\mathbb{Q}_p)$ (and hence of $E_2^s(\mathbb{Q}_p)$)

(1) (11.5) may not be exact;
(2) the projection ${}^t\widehat{A}^{\text{co-ord}}_{s/\mathbb{F}_p} \to {}^t\widehat{A}^{\text{co-ord}}_{r/\mathbb{F}_p}$ is purely inseparable of degree $\geq p^{s-r}$ (as the polarization of A_s has degree of high p-power $\geq p^s$).

These problems do not appear for the pull-back map $\widehat{A}_r^{\text{ord}} \xrightarrow{\sim} \widehat{A}_s^{\text{ord}}$ even over $\mathbb{F}_p$ as the exactness of $\widehat{A}_r^{\text{ord}} \hookrightarrow \widehat{J}_s^{\text{ord}} \twoheadrightarrow \alpha(\widehat{J}_s^{\text{ord}})$ is proven by the control of the Λ-adic BT group $\mathcal{G}$ in [H16, §17] and the projection $X_{s,0} \to X_{r,0}$ is étale outside the super-singular points for all s.

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