Galois Cohomology (Study Group)

1 Some P-adic Hodge Theory (by Chris Williams)

1.1 p-adic Hodge Theory

Aim: Study local Galois representation in the case $l = p$.

Notation. Let $K/\mathbb{Q}_p$ be a finite extension, $V/\mathbb{Q}_l$ a finite dimensional vector space with continuous action of $G_K := \mathrm{Gal}(\overline{K}/K)$

In the case $l \neq p$, the topologies are not compatible - there are not many representations and they are of algebraic nature.

In the case $l = p$, the topologies are compatible - we end up with far too many representations! So the study of them becomes p-adic analytic

Example. Let χ be the cyclotomic character. Define *weight space* $\mathcal{W} := \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\chi, \mathbb{C}_p^\chi) = \coprod_{p-1 \text{ copies}} \mathbb{Z}_p \supset \mathbb{Z}$. Then for all $s \in \mathcal{W}$, χ^s is a p-adic representations. We are only really "interested" in χ^s where $s \in \mathbb{Z}$.

Idea: Isolate "interesting" subcategories.

Fontaine's Strategy: Define *ring of periods*, i.e., topological $\mathbb{Q}_p$-algebra B with a continuous action of G_K. The idea is for some p-adic representation V, the invariant $D_B(V) := (V \otimes_{\mathbb{Q}_p} B)^{G_K}$.

Fact. *With stronger assumptions of B (G_K-regular) we have $\dim_{B^{G_k}} D_B(V) \leq \dim_{\mathbb{Q}_p} V$. (The stronger assumption will always be met in this section)*

We say that V is *B-admissible* if we have an equality.

Question: What are good choices of B?

Theorem. *There exists a ring of periods $\mathbb{B}_{\mathrm{dR}}$ such that*

1. *There is a natural filtration* $\mathrm{Fil}^i\mathbb{B}_{\mathrm{dR}}, i \in \mathbb{Z}$
2. *A p-adic representation V is $\mathbb{B}_{\mathrm{dR}}$-admissible if it "comes from geometry"*

We call such representation de Rham.

Example. Let E be an elliptic curve over $\mathbb{Q}_p$, $V_pE := T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then V_pE is de Rham

Let X be a proper, smooth variety over K, then $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ are de Rham.

Theorem. *There exists a ring of periods $\mathbb{B}_{\mathrm{crys}} \subset \mathbb{B}_{\mathrm{dR}}$ such that*

1. *There exists a natural Frobenius operator ϕ, and*
2. *A p-adic representation V is $\mathbb{B}_{\mathrm{crys}}$-admissible is* de Rham, *and representation that comes from geometry "with" good reduction at p are* $\mathbb{B}_{\mathrm{crys}}$-admissible.

We call such representation Crystalline.

Example. V_pE is crystalline if and only if E has good reduction at p.

Remark.

1. $\mathbb{B}_{\mathrm{dR}}$ and $\mathbb{B}_{\mathrm{crys}}$ are <u>huge</u> (in fact, they surject onto $\mathbb{C}_p = \widehat{\overline{K}}$)
2. $\mathbb{B}_{\mathrm{dR}}^{G_K} = K$ and $\mathbb{B}_{\mathrm{crys}}^{G_K} = K_0 =$ maximal unramified subfield of K.
3. We said $\mathbb{B}_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{crys}}$ had extra structure. This passes to $\mathbb{D}_{\mathrm{dR}}(V) := D_{\mathbb{B}_{\mathrm{dR}}}(V) = (V \otimes \mathbb{B}_{\mathrm{dR}})^{G_K}$ and $\mathbb{D}_{\mathrm{crys}}(V) := D_{\mathbb{B}_{\mathrm{crys}}}(V) = (V \otimes \mathbb{B}_{\mathrm{crys}})^{G_K}$.

1.2 Relation to Galois Cohomology

1.2.1 The group $H^1_*(K, V)$

Recall: Elements of $H^1(K, V)$ bijects with isomorphism classes of extension $0 \to V \to E \to \mathbb{Q}_p \to 0$ of the trivial representation by V. (Recap: given such an extension, we take the Galois cohomology, get $0 \to V^{G_K} \to E^{G_K} \to \mathbb{Q}_p \xrightarrow{\delta} H^1(K, V)$, and E correspond to $\delta(1) \in H^1(K, V)$.

Let V be de Rham. Then consider the complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & E & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V \otimes \mathbb{B}_{\mathrm{dR}} & \longrightarrow & E \otimes \mathbb{B}_{\mathrm{dR}} & \longrightarrow & \mathbb{B}_{\mathrm{dR}} & \longrightarrow & 0 \end{array}$$

We can take Galois cohomology:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^{G_K} & \longrightarrow & E^{G_K} & \longrightarrow & \mathbb{Q}_p & \xrightarrow{\delta} & H^1(K, V) \\ & & \downarrow & & \downarrow & & \downarrow \alpha & & \downarrow \beta \\ 0 & \longrightarrow & \mathbb{D}_{\mathrm{dR}}(V) & \longrightarrow & \mathbb{D}_{\mathrm{dR}}(E) & \longrightarrow & K & \xrightarrow[\gamma]{} & H^1(K, V \otimes \mathbb{B}_{\mathrm{dR}}) \end{array}$$

E is de Rham if and only if

$$\begin{aligned} \dim_K \mathbb{D}_{\mathrm{dR}}(E) &= \dim_{\mathbb{Q}_p}(E) \\ &= \dim_{\mathbb{Q}_p} V + 1 \\ &= \dim_K \mathbb{D}_{\mathrm{dR}}(V) + 1 \end{aligned}$$

if and only if $0 \to \mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{D}_{\mathrm{dR}}(E) \to K \to 0$ is exact, if and only if, γ is identically 0.

E correspond to $\phi = \delta(1) \in H^1(K, V)$. Now $\beta(\phi) = \beta\delta(1) = \gamma\alpha(1)$. So E is de Rham implies $\beta(\phi) = 0$. Conversely, if $\beta(\phi) = 0$, then $\delta\alpha(1) = 0$, but α is the inclusion, hence δ is identically 0.

So E is de Rham if it is in the kernel of β.

Definition. We set:

- $H^1_g(K, V) = \ker(H^1(K, V) \to H^1(K, V \otimes \mathbb{B}_{\mathrm{dR}}))$
- $H^1_f(K, V) = \ker(H^1(K, V) \to H^1(K, V \otimes \mathbb{B}_{\mathrm{crys}}))$
- $H^1_e(K, V) = \ker(H^1(K, V) \to H^1(K, V \otimes \mathbb{B}_{\mathrm{crys}}^{\phi=1}))$

Note. We have $H^1_g(K, V) \supset H^1_f(K, V) \supset H^1_e(K, V)$.

Proposition. *Let V be* de Rham *(respectively* crystalline*), and $0 \to V \to E \to \mathbb{Q}_p \to 0$ be an exact sequence corresponding to $\phi \in H^1(K, V)$. Then E is* de Rham *(respectively* crystalline*) if and only if $\phi \in H^1_g(K, V)$ (respectively in $H^1_f(K, V)$).*

1.2.2 Tate's Duality.

Recall: $V^* = \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$, and $V(1) = V \otimes \chi = V \otimes \varprojlim \mu_{p^n}$.

There exists a natural pairing $V \times V^*(1) \to \mathbb{Q}_p(1)$ given by $(v, \mu) \mapsto \mu(v)$. Under cup product, we get a perfect pairing, $H^i(K, V) \times H^{2-i}(K, V^*(1)) \to H^2(K, \mathbb{Q}_p(1)) \overset{\text{Tate}}{\cong} \mathbb{Q}_p$.

Theorem (Bloch - Kato)**.** *Under this pairing:*

- *$H^1_g(K, V)$ and $H^1_e(K, V^*(1))$ are exact annihilators*
- *$H^1_f(K, V)$ and $H^1_f(K, V^*(1))$ are exact annihilators*
- *$H^1_e(K, V)$ and $H^1_g(K, V^*(1))$ are exact annihilators.*

Example. For $V = V_pE$, $\dim H^1(K, V_pE) = 2$, by the above theorem, $\dim H^1_g = \dim H^1_f = \dim H^1_e = 1$. Hence $H^1_g = H^1_f = H^1_e$ in this case. (In fact, this happens for a large class of examples in which we are interested)

1.2.3 The exponential map

Fact. *There exists an exact sequence* $0 \to \mathbb{Q}_p \to \mathbb{B}_{\mathrm{crys}}^{\phi=1} \to \mathbb{B}_{\mathrm{dR}}/\mathrm{Fil}^0\mathbb{B}_{\mathrm{dR}} \to 0$.

If we tensor with V and then taking Galois cohomology, we get

$$0 \longrightarrow H^0(K, V) \longrightarrow \mathbb{D}_{\mathrm{crys}}^{\phi=1}(V) \longrightarrow \mathbb{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V) \xrightarrow{\exp} H^1(K, V) \longrightarrow H^1(K, V \otimes \mathbb{B}_{\mathrm{crys}}^{\phi=1})$$

Conclusion: We get a map $\exp : \mathbb{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}} \twoheadrightarrow H^1_e(K, V)$.

Remark. Usually, $\mathbb{D}_{\mathrm{crys}}^{\phi=1}(V)$ is trivial, which implies exp is an isomorphism.