Galois Cohomology (Study Group)

1 Group Cohomology (by Chris Birkbeck)

1.1 Definition of group cohomology

Let G be a group

Definition 1.1. A *G-module* M, is an abelian group with an action of G. I.e., there is an homomorphism $\theta : G \to \mathrm{Aut}(M)$. Usual, we take $g \in G, m \in M$ and define $\phi(g)m = g \cdot m$ with the properties

- $g(m_1 + m_2) = gm_1 + gm_2$
- $g(g'm) = (gg')m$

We can extend this action to an action of $\mathbb{Z}[G]$ on M.

Definition 1.2. For a G-module M, let $M^G = \{m \in M | gm = m \forall g \in G\}$

Fact. $M^G = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$. *We have a functor* $(-)^G$ *defined by* $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$.

Let A, B, C be G-modules with the short exact sequence $0 \to A \to B \to C \to 0$. Apply $(-)^G$ to get an exact sequence $0 \to A^G \to B^G \to C^G$.

Definition 1.3. Let G be as above, M a G-module. For $i \geq 0$, define $C^i(G, A)$ to be $\mathrm{Maps}(G^i, M)$ with $G^0 = \{1\}$.

Note that $C^0(G, M) \cong M$. We give $C^i(G, M)$ a group structure pointwise and $(g\phi)(x_1, \dots, x_n) = g(\phi(x_1, \dots, x_n))$.

Definition 1.4. We define *co-boundary homomorphism* $d_n : C^n(G, M) \to C^{n+1}(G, M)$ as follows. Let $f \in C^n(G, M)$, $g_i \in G$, then, $d_{-1} = 0$ and for $n \geq 0$

$$\begin{aligned} d_n(f)(g_1, \dots, g_{n+1}) &= g_1 f(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2} \dots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \dots, g_n) \end{aligned}$$

$n = 0$ $\qquad d_0 f(g) = gf - f$

$n = 1$ $\qquad d_1(f)(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$

Exercise: Show that $0 \xrightarrow{d_{-1}} C^0(G, M) \xrightarrow{d_0} C^1(G, M) \xrightarrow{d_1} \dots$ is a cochain complex.

Let $Z^n(G, M) = \ker d_n$. These are called *n-cocycles*

Let $B^n(G, M) = \mathrm{im}\, d_{n-1}$. These are called *$n$-coboundaries.*

Define the nth *Cohomology group* to be

$$H^n(G, M) = \frac{Z^n(G, M)}{B^n(G, M)} \text{ for } n \geq 0$$

$n = 0$ $\qquad$ We have $B^0(G, M) = \{0\}$, $Z^0(G, M) = \{f \in C^0(G, M) | gf = f \forall g \in G\}$, hence $H^0(G, M) = M^G$

$n = 1$ $\qquad$ We have $B^1(G, M) = \{f | f(g) = ga - a \text{ for some } a \in M\}$, and $Z^1(G, M) = \{f | f(g_1, g_2) = g_1 f(g_2) + f(g_1)\}$, and $H^1(G, M) = Z^1/B^1$. If G acts trivially then $H^1 = \mathrm{Hom}(G, M)$.

1.2 Topological groups

Let G be a topological (profinite group) and T a topological G-module (i.e., T is an topological abelian group with continuous G-action)

Definition 1.5. We define $C^i_{\text{cts}}(G,T) = \text{CtsMaps}(G^i,T)$.

Take T,T',T'' to be topological G-modules and let $0 \to T' \to T \to T'' \to 0$ be a short exact sequence. Furthermore let $S : T'' \to T$ be a continuous section (as spaces). We get $0 \to C^i_{\text{cts}}(G,T') \to C^i_{\text{cts}}(G,T) \to C^i_{\text{cts}}(G,T'') \to 0$ is exact for all $i \geq 0$. We define $d_n : C^n_{\text{cts}}(G,T) \to C^{n+1}_{\text{cts}}(G,T)$ as before and then we get a long exact sequence

$$\cdots \to H^n_{\text{cts}}(G,T') \to H^n_{\text{cts}}(G,T) \to H^n_{\text{cts}}(G,T'') \to H^{n+1}_{\text{cts}}(G,T') \to \dots$$

1.3 Maps between cohomology groups

Definition 1.6. Let G,G' be groups and M (respectively M') be a G-module (respectively G'-module). We say $\phi : G' \to G$ and $\psi : M \to M'$ are *compatible* if for $g' \in G'$, $a \in M$, $\psi(\phi(g')a) = g'\psi(a)$.

These induces a map $C^i(G,M) \to C^i(G',M')$ which in turn give a map $H^i(G,M) \to H^i(G',M')$

Example.

- Let $H \leq G$, $\phi : H \hookrightarrow G$ and $\psi = \text{id} : M \to M$. We get (*restriction*) $\text{Res} : H^n(G,M) \to H^n(H,M)$ defined by $[f] \mapsto [f|_H]$.
- Let $H \leq G$, $\phi : G \to G/H$, $\psi : M^H \hookrightarrow M$. Then we get (*inflation*) $\text{Inf} : H^n(G/H,M^H) \to H^n(G,M)$ defined by $\inf(f)(g) = f(\overline{g})$ for $f \in Z^n(G/H,M^H)$.
- We have an exact sequence

$$0 \to H^1(G/H,M^H) \overset{\inf}{\to} H^1(G,M) \overset{\text{res}}{\to} H^1(H,M)^{G/H} \overset{\text{trans}}{\to} H^2(G/H,M^H) \overset{\inf}{\to} H^2(G,M) \to$$

1.4 H^1 extension

Let A,B be G-modules, give $A \otimes_{\mathbb{Z}} B$ a G-action. There exists a family of maps $C^n(G,A) \otimes C^n(G,B) \overset{\cup}{\to} C^{n+m}(G,A \otimes B)$. For $n = 0$, $A^G \otimes B^G \to (A \otimes B)^G$. This will extend to a map on H^n.

We also get $d(f \cup g) = df \cup g + (-1)^n(f \cup dg)$.