Galois Cohomology (Study Group)

1 Further properties of group cohomology; relations to topological cohomology (by Matthew Spencer)

1.1 Homology Topological space

Let X be a topological space, a singular complex, $\sigma : \Delta^n \to X$. We form $C_n(X)$, which has elements of the form $\sum n_i \sigma_i$. We define $d_n\sigma = \sum (-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]} \in C_{n-1}(X)$. We get a chain

$$C_n(X) \to C_{n-1}(X) \to C_{n-2}(X) \to \cdots \to C_0(X) \to 0$$

We can check that $d^2 = 0$. We define

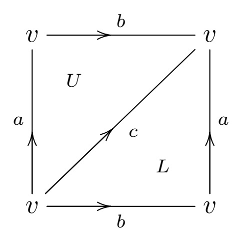
$$H_n(X) = \frac{\ker(d_n)}{\operatorname{im}(d_{n+1})}$$

Example. If X is path connected we have that $H_0(X) \cong \mathbb{Z}$.

We have that a map $f : X \to Y$ induce maps on homology.

Fact. *If $f : X \to Y$ and $g : X \to Y$ are homotopic maps then the maps $H_n(X) \to H_n(Y)$ induces by f and g agree for all n.*

Example.



We have the sequence

$$0 \xrightarrow{d_3} C_2(X) \xrightarrow{d_2} C_1(X) \xrightarrow{d_1} C_0(X) \xrightarrow{d_0} 0$$

$$= \langle U, L\rangle \qquad = \langle a, b, c\rangle \qquad = \langle v\rangle$$

We see that $\operatorname{im} d_2 = a + b - c$. We have

$$H_1(T) \cong \langle a, b, a+b-c\rangle \,/\, \langle a+b-c\rangle \cong \langle a, b\rangle$$

1.1.1 Cohomology

We take a pair X, A where X is a topological space and A an Abelian group. We take the sequence

$$C_n(X) \to C_{n-1}(X) \to C_{n-2}(X) \to \cdots \to C_0(X) \to 0$$

and apply $\mathrm{Hom}_{\mathbb{Z}}(-, A)$ to it, to get another chain which we denote $C^n(X, A)$. We define the map δ between them by $\delta f(a) = f(da)$, again we see $\delta^2 = 0$. Again we define $H^n(X, A) = \ker \delta^n / \mathrm{im}(\delta^{n-1})$.

Universal Coefficient Theorem. $H^n(X, A) \cong \mathrm{Hom}(H_n(X), A) \oplus \mathit{Ext}_{\mathbb{Z}}(H_{n-1}(X), A)$

1.1.2 Cup product

Let A be a ring and let $\phi \in C^k(X, A)$ and $\psi \in C^l(X, A)$. We define the cup product as

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma|_{[v_{k+1},\ldots,v_{k+l}]}) \in C^{k+l}(X, A)$$

We have

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^k \phi \cup \delta\psi$$

So we have a well defined map $H^k(X, A) \times H^l(X, A) \to H^{k+l}(X, A)$ defined as $[\phi] \times [\psi] \mapsto [\phi \cup \psi]$.

Let $H^*(X, A) = \oplus H^n(X, A)$, this is (by the previous map) a graded ring. Note that $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$.

1.1.3 Cap product

Let $n \geq p$, $\phi \in C^p(X, A)$ and $\sigma \in C_n(X, A)$ (note that $C_n(X, A)$ is taken by considering $C_n(X)$ and applying $- \otimes_{\mathbb{Z}} A$ to it). We define the cap product as

$$\sigma \cap \phi = \phi(\sigma|_{[v_0,\ldots,v_p]}) \sigma_{[v_{p+1},\ldots,v_n]} \in C_{n-p}(X, A)$$

We have

$$d(\sigma \cap \phi) = (-1)^p (d\sigma \cap \phi - \sigma \cap \delta\phi)$$

Let K be a commutative ring with unit. Using the above, we have the map

$$\cap : H_n(X, K) \times H^p(X, K) \to H_{n-p}(X, K)$$

Theorem 1.1. *Let M be a compact manifold without boundary, K-orientable with fundamental class $[m] \in H_n(M, K)$. The map $D : H^k(M, K) \to H_{n-k}(M, K)$ defined by $D(\alpha) = [m] \cap \alpha$ is an isomorphism.*

1.2 Group Cohomology

Let G be a discrete group. Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$, let M be a $\mathbb{Z}[G]$-module. Consider the resolution

$$F_i \to F_{i-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$$

We want to apply $(- \otimes_{\mathbb{Z}} M)_G = (F_i \otimes M)/ \mod G-$ action, to it and get another chain sequence. We define $H_n(G, M)$ to be the homology of that chain. If instead we apply $\mathrm{Hom}_{\mathbb{Z}[G]}(F_i, M)$, then we call its homology $H^n(G, M)$.

We define G-complex. A CW-complex with a G-action which respects CW-complex structure. We say that this is free if G acts freely. If X is contractible, then $H_n(X) = H_n\{\mathrm{pt}\}$. In particular

$$C_n(X) \to C_{n-1}(X) \to \cdots \to C_0(X) \to \mathbb{Z} \to 0$$

is a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-module.

There exists a space $k(G, 1) =: Y$ such that

1. Y is connected

2. $\pi_1(Y) = G$

3. If X is the universal cover, X is contractible.

Fact. *If X is a free G-complex, Y the orbit X/G, then $C_*(Y) = C_*(X)_G$, where $(-)_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} -$.*

This gives $H_*(G) = H_*(Y)$.

Let us now assume that G acts trivially on M, then $H_n(Y, M) = H_n(G, M)$ and $H^n(Y, M) = H^n(G, M)$.

If M has non-trivial G action, then $H^n(G, M) = H^n(Y, \mathcal{M})$ and $H_n(G, M) = H_n(Y, \mathcal{M})$ where $\mathcal{M}$ is a local coefficient system on $k(G, 1)$.

$k(G, 1)$ is an example of a classifying space for G, we call it BG.

Fact. *If $H \leq G$, we have a mapBH $\to BG$. This induce a map on the cohomology $(res^G_H) : H^n(G, A) \to H^n(H, A)$*

This is motivation for the next sentence: Suppose if we have a finite map $f : X \to Y$, where this time X and Y are manifolds with dimension n

$$\begin{array}{ccc} H^k(X) & \longleftarrow & H^k(Y) \\ \| & & \| \\ H_{n-k}(X) & \longrightarrow & H_{n-k}(Y) \end{array}$$

Now let $H \leq G$ with $|G : H| < \infty$, we have $\mathrm{cor}^H_G : H^n(H, A) \to H^n(G, A)$ defined by

$$(\mathrm{cor}x)(\sigma_0, \ldots, \sigma_n) = \sum_{c \in H \backslash G} \bar{c}^{-1} x(\bar{c}\sigma_0 \overline{(c\sigma_0)}^{-1}, \ldots, \bar{c}\sigma_n \overline{(c\sigma_n)}^{-1})$$

We get the identity

$$\mathrm{cor}(\alpha \cup \mathrm{res}\beta) = (\mathrm{cor}\alpha) \cup \beta$$