Galois Cohomology (Study Group)

1 Cohomology of Arithmetic Groups and Eichler - Shimura Isomorphism

In nice cases we can define an arithmetic group as follow. Let K be a number field. "An *arithmetic group* is a subgroup of $G(K)$ that is commensurable with $G(\mathcal{O}_K)$. (Where G is the general linear group)"

Remark. Γ_1 and Γ_2 are commensurable if $\Gamma_1 \cap \Gamma_2$ have finite index in Γ_1 and Γ_2. This is an equivalence relation

Example. $\mathrm{GL}_n(\mathcal{O}_K)$, $\mathrm{SL}_n(\mathcal{O}_K)$

In particular, we will work with $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{PSL}_2(\mathbb{Z})$ and finite index subgroup of them.
Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be of finite index. We want to have a Γ-module so that we can consider cohomology.

Definition 1.1. $V_k(\mathbb{C}) = \{\text{homogenous degree } k \text{ polynomial in } \mathbb{C}[X,Y]\}$. We will say that this has "weight" $k+2$.

This has a Γ-action defined by: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$P|_\gamma(X,Y) = P(aX + cY, bX + dY)$$

Aside: You can define the other action by $P|_\gamma(X,Y) = P(dX - cY, -bX + aY)$

Note. $V_0(\mathbb{C}) \cong \mathbb{C}$

Fact. *For $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ of finite index we have that $H^i(\Gamma, V_k(\mathbb{C})) = 0$ for all $i \geq 2$. (This use M-V and the fact that $\mathrm{SL}_2(\mathbb{Z}) \cong C_4 *_{C_2} C_6$)*

So we are interested in $H^1(\Gamma, V_k(\mathbb{C}))$. Recall the space $M_k(\Gamma) = \{f : \mathcal{H} \to \mathbb{C} : (f|_k\gamma)(z) = (cz+d)^k f(\gamma z) = f(z) \forall \gamma \in \Gamma\}$ and $S_k(\Gamma) \subset M_k(\Gamma)$.

Definition 1.2. The space of *antiholomorphic cusp form* $\overline{S_k(\Gamma)}$ consists of functions $\overline{f}(z) := \overline{f(z)}$ with $f \in S_k(\Gamma)$

Eichler - Shimura Isomorphism. *Let $k \geq 2$ and $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be of finite index. We have the map*

$$M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \to H^1(\Gamma, V_{k-2}(\mathbb{C})) \ \ (\dagger)$$

is an isomorphism.

To define the map, we need to introduce some more notation. Fix $z_0 \in H$, let $f \in M_k(\Gamma)$ with $k \geq 2$ and $g, h \in \mathrm{SL}_2(\mathbb{Z})$. Define

$$\begin{aligned} I_f(gz_0, hz_0) &:= \int_{gz_0}^{hz_0} f(z)(Xz+Y)^{k-2} dz \in V_{k-2}(\mathbb{C}) \\ I_{\overline{f}}(gz_0, hz_0) &:= \int_{gz_0}^{hz_0} \overline{f(z)}(Xz+Y)^{k-2} dz \in V_{k-2}(\mathbb{C}) \end{aligned}$$

The map $(\dagger)$ is defined as

$$(f, \overline{g}) \mapsto (\gamma \mapsto I_f(z_0, \gamma z_0) + I_{\overline{g}}(z_1, \gamma z_1))$$

One needs to check that this map is independent of the choice of z_0 and z_1 (and it is in fact an isomorphism)

Definition 1.3. We define the parabolic cohomology group (or cusp cohomology) via the kernel of the restriction map in

$$0 \to H^1_{\text{par}}(\Gamma, V_k(\mathbb{C})) \to H^1(\Gamma, V_k(\mathbb{C})) \overset{\text{res}}{\to} \prod_{c \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})} H^1(\Gamma_c, V_k(\mathbb{C}))$$

where Γ_c is the stabilizer of the cusp c in Γ.

In weight 2 we can do this topologically. Let $Y_\Gamma = \Gamma \backslash \mathcal{H}$, then $Y_\Gamma = k(\Gamma, 1)$ (as described in Matthew's talk). Hence $H^1(\Gamma, \mathbb{C}) \cong H^1(Y_\Gamma, \mathbb{C})$, but we are actually interested in the Borel - Serre compactification of Y_Γ, call itX_Γ. Then $H^1(Y_\Gamma, \mathbb{C}) \cong H^1(X_\Gamma, \mathbb{C})$ and we define the cusp cohomology as

$$0 \to H^1_{\text{cusp}}(X_\Gamma, \mathbb{C}) \to H^1(X_\Gamma, \mathbb{C}) \overset{\text{res}}{\to} H^1(\partial X_\Gamma, \mathbb{C})$$

Hence topologically, $H^1_{\text{cusp}}(\Gamma, \mathbb{C}) = \{f : \Gamma \to \mathbb{C} | f \text{ vanishes at every cusp}\}$

Proposition 1.4. *The Kernel of the composition of the Eichler - Shimura map with the restriction map*

$$M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \to H^1(\Gamma, V_{k-2}(\mathbb{C})) \to \prod_{c \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})} H^1(\Gamma_c, V_{k-2}(\mathbb{C}))$$

is equal to $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$

Corollary 1.5. *If* $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ *has finite index, then* $H^1_{\text{par}}(\Gamma, V_k(\mathbb{C})) \cong S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$

1.0.1 Hecke Operators

We now restrict ourselves to $\Gamma_0(N) = \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$ and $\Gamma_1(N) = \left\{ A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right.$
We need to define the following objects:

- $\Delta_0^n(N) = \{A \in M_2(\mathbb{Z}) : A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N, \det A = n, \gcd(a, n) = 1\}$
- $\Delta_1^n(N) = \{A \in M_2(\mathbb{Z}) : A \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \mod N, \det A = n\}$
- $\Delta_0(N) = \cup_{n \in \mathbb{N}} \Delta_0^n$
- $\Delta_1(N) = \cup_{n \in \mathbb{N}} \Delta_1^n$

From now on Δ, Γ will refer to either the pair $\Delta_1(N), \Gamma_1(N)$ or $\Delta_2(N), \Gamma_2(N)$.

Definition 1.6. Let $\alpha \in \Gamma$, define $\Gamma_\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha$ and $\Gamma^\alpha = \Gamma \cap \alpha\Gamma\alpha^{-1}$

Note. $\Gamma, \alpha^{-1}\Gamma\alpha$ and $\alpha\Gamma\alpha^{-1}$ are all pairwise commensurable

Notation. If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we let $\alpha^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \det(\alpha) \cdot \alpha^{-1}$

Definition 1.7. Let $\alpha \in \Delta$. The Hecke operator τ_α on a group cohomology is the composite

$$\begin{array}{ccc} H^1(\Gamma, V_k(\mathbb{C})) & \overset{\tau_\alpha}{\dashrightarrow} & H^1(\Gamma_1, V_k(\mathbb{C})) \\ {\scriptstyle \text{res}} \downarrow & & \uparrow {\scriptstyle \text{cores}} \\ H^1(\Gamma^\alpha, V_k(\mathbb{C})) & \underset{\text{conj}_\alpha}{\longrightarrow} & H^1(\Gamma_\alpha, V_k(\mathbb{C})) \end{array}$$

where conj_α is defined by $c \mapsto (g_\alpha \mapsto \alpha^\iota \cdot c(\alpha g_\alpha \alpha^{-1})$

We can in fact explicitly write down how τ_α acts on H^1.

Proposition 1.8. *Let $\alpha \in \Delta$, and suppose that $\Gamma\alpha\Gamma = \sqcup_{i=1}^n \Gamma\beta_i$ (which can always be done). Then τ_α acts on H^1 and H^1_{cups} by sending non-homogeneous cocycle c to $\tau_\alpha c$ which is defined by*

$$(\tau_\alpha c)(g) = \sum_{i=1}^n \beta_i c(\beta_i g \beta_{\sigma_g(i)}^{-1})$$

where $\sigma_g(i)$ is such that $\beta_i g \beta_{\sigma_g(i)}^{-1} \in \Gamma$.

Definition 1.9. For a positive integer n, the Hecke operator T_n is defined as

$$\sum_{\alpha \in \Gamma \backslash \Delta^n / \Gamma} \tau_\alpha$$

Note. If p is a prime not dividing N, then T_p is τ_α for $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$.

Definition 1.10. We can also define the diamond operator for a coprime to N, $\langle a \rangle$, as τ_α where α is such that $\alpha \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \mod N$

Proposition 1.11. *The Eichler - Shimura isomorphism is compatible with Hecke operators*

In particular we have $S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \cong H^1_{\text{cusp}}(\Gamma, V_{k-2}(\mathbb{C}))$ as Hecke modules. Note that $H^1_{\text{cusp}}(\Gamma, V_k(\mathbb{Z})) \subset H^1_{\text{cusp}}(\Gamma, V_k(\mathbb{C}))$ and that (as the defining diagram make sense), it is stable under the Hecke operators. That is, we have a lattice (H^1/Tor) in $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$ that is stable under the Hecke operators, so after fixing a basis, we have that a Hecke operator acts $T : \mathbb{Z}^d \to \mathbb{Z}^d$. Hence the characteristic polynomial of a Hecke operator is integral.

Let $\chi : (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a Dirichlet character. Then if a $\mathbb{C}$-vector space V admits an action of $(\mathbb{Z}/n\mathbb{Z})^*$ we define

$$V^\chi = \{v \in V : g \cdot v = \chi(g) \cdot v \forall g \in (\mathbb{Z}/n\mathbb{Z})^*\}$$

We define $M_k(\Gamma_1(N), \chi)$ as the χ-eigenspace $M_k(\Gamma_1(N))^\chi$.

Theorem 1.12 (Eichler - Shimura)**.** *Let $N \geq 1$, $k \geq 2$ and $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ be a Dirichlet character. Then the Eichler - Shimura map gives an isomorphism*

$$\begin{aligned} M_k(\Gamma_1(N), \chi) \oplus \overline{S_k(\Gamma_1(N), \chi)} &\cong H^1(\Gamma_0(N), V^\chi_{k-2}(\mathbb{C})) \\ S_k(\Gamma_1(N), \chi) \oplus \overline{S_k(\Gamma_1(N), \chi)} &\cong H^1_{\text{cusp}}(\Gamma_0(N), V^\chi_{k-2}(\mathbb{C})) \end{aligned}$$