Galois Cohomology (Study Group)

1 Galois Cohomology and applications (Angelos Koutsianas)

1.1 Tate's Theorem

Theorem 1.1. *Suppose $i > 0$ and $T = \varprojlim_n T_n$ where each T_n is a finite (discrete) G-module. If $H^{i-1}(G, T_n)$ is finite for all n, then $H^i(G,T) = \varprojlim_n H^i(G, T_n)$*

Theorem 1.2. *If T is a finitely-generated $\mathbb{Z}_p$-module, then for every $i \geq 0$ $H^i(G,T)$ has no divisible elements and $H^i(G,T) \otimes \mathbb{Q}_p \xrightarrow{\sim} H^i(G, T \otimes \mathbb{Q}_p)$.*

Principle: "If G satisfies the condition that $H^i(G,M)$ is finite for finite M, we have nice theorems"

1.2 Hilbert's 90, Kummer Theorem and more.

Let $K \subset L$ be field extensions such that L/K is Galois, and denote $G_{L/K} := \mathrm{Gal}(L/K)$. Then $G_{L/K}$ is profinite.

$$H^i(G_{L/K}, L^*) \cong \varinjlim_{L \supset M \supset K, \text{finite, Galois}} H^i(G_{M/K}, M^*).$$

Theorem 1.3 (Hilbert's 90)**.** *We have $H^1(G_{L/K}, L^*) = 1$.*
General case: $H^1(G_{L/K}, \mathrm{GL}_n(L)) = 1$.

Let us assume $\overline{K}$ is separable. We have the following short exact sequence

$$1 \longrightarrow \mu_N \longrightarrow \overline{K}^* \xrightarrow{N} \overline{K}^* \longrightarrow 1$$

where μ_N is the group which are N-th root of unity. We assume $\mu_N \subseteq K^*$. We get

$$1 \longrightarrow \mu_N \longrightarrow K^* \xrightarrow{N} K^* \xrightarrow{\delta} H^1(G_{\overline{K}/K}, \mu_N) \longrightarrow H^1(G_{\overline{K}/K}, \overline{K}^*) \longrightarrow \dots$$

Since $H^1(G_{\overline{K}/K}, \overline{K}^*) = 1$ (by Hilbert's 90), we have that δ is surjective. Hence we get:

Theorem 1.4 (Kummer)**.** $\mathrm{Hom}_{\mathrm{ctn}}(G_{\overline{K}/K}, \mu_n) = H^1(G_{\overline{K}/K}, \mu_N) \cong K^* / (K^*)^N$

Definition 1.5. Let p be a prime then $\mathbb{Z}_p(1) := \varprojlim_n \mu_{p^n}$

Since $H^0(G_{\overline{K}/K}, \mu_{p^n}) = \mu_{p^n} \cap K < \infty$ for all $n \in \mathbb{N}$, we can use Tate's theorem to get $H^1(G_{\overline{K}/K}, \mathbb{Z}_p(1)) \cong K^* \otimes_{\mathbb{Z}} \mathbb{Z}_p$.
Let E/K be an elliptic curve, with K a number field. We have the following short exact sequence

$$0 \longrightarrow E[m] \longrightarrow E(\overline{K}) \xrightarrow{m} E(\overline{K}) \longrightarrow 0 \,.$$

This gives us the following long exact sequence

$$0 \longrightarrow E(K)[m] \longrightarrow E(K) \xrightarrow{m} E(K) \xrightarrow{\delta} H^1(G_{\overline{K}/K}, E[m]) \longrightarrow H^1(G_{\overline{K}/K}, E(\overline{K})) \longrightarrow \dots$$

$$0 \longrightarrow E(K)/mE(K) \xrightarrow{\delta} H^1(G_{\overline{K}/K}, E[m]) \longrightarrow H^1(G_{\overline{K}/K}, E(\overline{K}))[m] \longrightarrow 0$$

Again by Tate we have

$$E(K) \otimes \mathbb{Z}_p = \varprojlim_n \frac{E(K)}{p^n E(K)} \overset{\delta}{\hookrightarrow} H^1(G_{\overline{K}/K}, T_p(E))$$

where $T_p(E) := \varprojlim_n E[p^n]$.

1.3 Milnor K-group

Let K be a local field, the Hilbert symbol is a bilinear function $K^* \times K^* \to \mu_n$ such that $(a, 1-a) = 1$ when $a, 1-a \in K^*$.

$n = 2$ In this case the Hilbert symbol is defined as $(a,b) = \begin{cases} 1 & z^2 = ax^2 + by^2 \,\text{has non trivial solution on}\, K^3 \\ -1 & \text{else} \end{cases}$.

Definition 1.6. We define the n-th Milnor K-group of the field F (for $n \geq 1$) to be

$$K_n^M(F) = \overbrace{(F^* \otimes \cdots \otimes F^*)}^{n\,\text{times}}/F_n$$

where

$$F_n = \langle a_1 \otimes \cdots \otimes a_n : \exists i \neq j \,\text{with}\, a_i + a_j = 1 \rangle .$$

We have the following map $\underbrace{F^* \times \cdots \times F^*}_{n} \to K_n^M(F)$ defined by $(a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\} := a_1 \otimes \cdots \otimes a_n$ mod F_n. Observing that $F_n \otimes \overbrace{F \otimes \cdots \otimes F}^{m}$ and $\overbrace{F \otimes \cdots \otimes F}^{m} \otimes F_n$ are both in F_{n+m}, we can define $K_n^M(F) \times K_m^M(F) \to K_{n+m}^M(F)$ by $(\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_m\}) \to \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$. Hence we have a graded ring $K^M(F) = \oplus_{n \geq 0} K_n^M(F)$ where we define $K_0^n(F) = \mathbb{Z}$.

We have a short exact sequence

$$1 \longrightarrow \mu_N \longrightarrow \overline{F}^* \xrightarrow{N} \overline{F}^* \longrightarrow 1$$

$$\delta_F : F^* \to H^1(G_{\overline{F}/F}, \mu_N)$$

Recall that we have the cup product:

$$\underbrace{H^1(G_{\overline{F}/F}, \mu_N) \times \cdots \times H^1(G_{\overline{F}/F}, \mu_N)}_{n} \overset{\cup}{\to} H^n(G_{\overline{F}/F}, \mu_n^{\otimes n})$$

Theorem 1.7. *The map $\cup \circ \delta$ induces a homomorphism $h_F : K_n^M(F) \to H^n(G_{\overline{F}/F}, \mu_N^{\otimes n})$.*

Bloch - Kato - Voevodsky's Theorem (Fields Medal). *For every field F and $(N, \mathrm{char} F) = 1$, then h_F gives an isomorphism*

$$h_F : K_n^M(F)/N K_n^M(F) \overset{\sim}{\to} H^n(G_{\overline{F}/F}, \mu_N^{\otimes n})$$

for all $n \geq 1$.