Special values of Rankin-Selberg convolution L-functions

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M4K9 Dissertation

Submitted to The University of Warwick

Mathematics Institute

April, 2016

Contents

1 Introduction

Number theory has been around for millennia, and as such, it is one of the most intricate branches of mathematics. It is also a very active area of research. Over the last couple of centuries, several powerful tools have been developed, such as the theories of modular forms and of L-functions, and their study is far from over. One of the areas of research is the investigation of special values of L-functions. In this project, we will look at some particular examples of approaching this topic. Following (for the most part) the work of Shimura in the first half, and Zagier in the second, we will see what happens at some integral values of the Rankin-Selberg convolution L-function associated to a pair of modular forms.

Painting a complete picture of how these results fit in the greater landscape of research is far beyond the scope of this project. Instead, the project can be viewed as an attempt to create a "shortest path" from the material covered in lectures to a reasonable understanding of the results and of the methods used to prove them. In this spirit, we will often introduce various definitions and techniques *ad hoc*, and only discuss the properties immediately useful to us. Without further ado, let us start.

2 Special values of the convolution L-function of two forms of different weights

In this section, we look at the Rankin-Selberg convolution of L-functions associated to forms of different weights. In particular, we will study the rationality of some expressions involving the values of the convolution at integers inside the critical strip.

Definition 1. *Throughout this section, let*

$$f = \sum_{n=1}^{\infty} a_n q^n, \qquad g = \sum_{n=0}^{\infty} b_n q^n$$

be modular forms of level 1 and weights k and l respectively, with $k > l$. In particular, we *want f to be a cusp form. Their associated L-functions are*

$$L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad L(g,s) = \sum_{n=0}^{\infty} b_n n^{-s}.$$

We define their Rankin-Selberg convolution L-function *to be*

$$L(f \otimes g, s) = \zeta(2s-k-l+2) \sum_{n=1}^{\infty} a_n b_n n^{-s}.$$

Also recall the definition of the Petersson product:

$$\langle f,g \rangle = m(\Phi)^{-1} \int_{\Phi} \overline{f(z)} g(z) y^{k-2}\, dx\, dy,$$

where $m(\Phi)$ is the measure of the fundamental domain with respect to $y^{-2}\, dx\, dy$. Note that we don't need holomorphy of f and g for this to make sense: we can extend the definition to all continuous functions satisfying the same automorphic property as f and g, provided the integral converges.

2.1 The general idea

The main result that we are going to study is a certain rationality relation between special values of the convolution function and the Petersson norm of f whenever f is a normalised Hecke eigenform, shown by Shimura in [Sh1]. Its statement is given in theorem 3 of the paper, and in our notation, it says that for m with $\frac{1}{2}(k+l-2) < m < k$, the values $\pi^{l-2-2m}\langle f,f\rangle^{-1} L(f \otimes g, m)$ lie in the extension of the rationals by the coefficients of f and g [1]. The main idea behind the proof is as follows: first, we find an integral formula for the convolution. This is done by using an unfolding method introduced by Rankin in [Ran]. The formula expresses the convolution as the integral of $\overline{f}g$ times a non-holomorphic Eisenstein series, which leads us to study the spaces of nearly holomorphic modular forms. Each special value of the convolution is thus expressed as the Petersson product of f with such a form. Finally, Shimura shows that the interaction of the Petersson product with those forms is "nice": we can find an orthogonal basis containing f of a space in which that form lies, and that allows us to relate the product to the norm of f using simple linear algebra.

Shimura does the whole procedure in general level, but that makes the notation cumbersome, and poses a few extra technical difficulties without bringing much additional insight, so we will restrict ourselves to the level 1 case.

[1] In [Sh2], the statement is extended to all values between k and l by writing a functional equation involving the non holomorphic Eisenstein series.

2.2 The unfolding method

Using Shimura's notation, write

$$D(s,f,g)=\sum_{n=1}^{\infty}a_nb_nn^{-s}.$$

Put $f_\rho(z)=\overline{f(-\overline{z})}=\sum\limits_{n=1}^{\infty}\overline{a_n}q^n$, and recall the definition of the gamma function,

$$\Gamma(s)=\int_0^\infty u^{s-1}e^{-u}du.$$

By Parseval's theorem[2], we have

$$\int_0^1\overline{f_\rho(z)}g(z)\,dx=\sum_{n=1}^{\infty}a_nb_ne^{-4\pi ny},\qquad z=x+iy.$$

Change variables in the expression for $\Gamma(s)$ by setting $u=4\pi ny$ (this can be done for any given $n\in\mathbb{N}$) to get

$$\Gamma(s)=(4\pi n)^s\int_0^\infty y^{s-1}e^{-4\pi ny}\,dy,$$

then multiply both sides by $(4\pi)^{-s}D(s,f,g)$:

$$\begin{aligned}(4\pi)^{-s}\Gamma(s)D(s,f,g)&=\sum_{n=1}^{\infty}a_nb_nn^{-s}n^s\int_0^\infty y^{s-1}e^{-4\pi ny}\,dy\\&=\int_0^\infty y^{s-1}\sum_{n=1}^{\infty}a_nb_ne^{-4\pi ny}\,dy\\&=\int_0^\infty y^{s-1}\int_0^1\overline{f_\rho}g\,dx\,dy\end{aligned}\tag{1}$$

Here is where the unfolding argument comes in: on the right hand side, we have an integral over the vertical strip $[0,1]\times[0,\infty)$. We can cover this strip with translates of the fundamental domain for the modular group. This allows us to rewrite the expression as an integral (over the fundamental domain) of an infinite sum indexed by the translates. To see what happens to the integrand, we use the automorphic property of f and g: for $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$, we have

$$(\overline{f_\rho}gy^{s+1})\circ\gamma=(cz+d)^{l-k}|cz+d|^{2k-2-2s}\overline{f_\rho}gy^{s+1}.$$

[2]This is a standard theorem in Fourier analysis; here, we apply it to $f_\rho g$ as a function of x.

Let R be a set of representatives for $SL_2(\mathbb{Z})_\infty \backslash SL_2(\mathbb{Z})$, and define the Eisenstein series

$$E_\lambda(z,s) = \sum_{\gamma \in R} (cz+d)^{-\lambda} |cz+d|^{-2s} = \sum_{\substack{c,d \\ (c,d)=1}} (cz+d)^{-\lambda} |cz+d|^{-2s} \tag{2}$$

(Shimura uses E^*_λ instead of E_λ). We will simply write $E_\lambda(z)$ for the value of E_λ at $s = 0$. Recalling that the measure $y^{-2} dx\, dy$ is invariant under the action of the modular group, we get:

$$\begin{aligned}
(4\pi)^{-s}\Gamma(s)D(s,f,g) &= \int_0^\infty \int_0^1 y^{s+1} \overline{f_\rho} g \ y^{-2} dx\, dy \\
&= \sum_{\gamma \in R} \int_{\gamma\Phi} y^{s+1} \overline{f_\rho} g \ y^{-2} dx\, dy \\
&= \sum_{\gamma \in R} \int_{\Phi} (\overline{f_\rho} g y^{s+1}) \circ \gamma \ y^{-2} dx\, dy \\
&= \int_\Phi \sum_{\gamma \in R} (cz+d)^{l-k} |cz+d|^{2k-2-2s} \overline{f_\rho} g y^{s+1} \ y^{-2} dx\, dy \\
&= \int_\Phi \overline{f_\rho} g \cdot E_{k-l}(z, s+1-k) y^{s+1} \ y^{-2} dx\, dy \\
&= \int_\Phi \overline{f_\rho} g \cdot y^{s+1-k} E_{k-l}(z, s+1-k) y^{k-2} \, dx\, dy
\end{aligned} \tag{3}$$

In the last line, we have written $y^{s+1} = y^{s+1-k} y^k$. The expression now looks (formally) like the Petersson product of f with $g \cdot y^{s+1-k} E_\lambda(z, s+1-k)$. Let us look more closely at functions $y^m E_\lambda(z,m)$ with $\lambda, m \in \mathbb{Z}$ and $\lambda \geq 2$. This expression converges absolutely for $\mathcal{R}(2m) > 2 - \lambda$. It is only holomorphic when $m = 0$, but it's easy to see that it still satisfies the same automorphic property as modular forms of weight λ. It follows that $g \cdot y^m E_\lambda(z,m)$ will transform like a modular form of weight $l + \lambda$, so in fact the Petersson product above makes sense. In the next section, we will study the so called nearly holomorphic modular forms (of which $y^m E_\lambda(z,m)$, for a suitably chosen m, is an example).

2.3 Nearly holomorphic modular forms and the δ_k operator

Definition 2. *A* nearly holomorphic modular form *of weight k is a continuous function h with the following properties:*

(i) h transforms like a modular form of weight k (i.e. $h(z+1) = h(z)$ and $h(-1/z) = z^k h(z)$).

(ii) $h(z) = \sum_{\nu=0}^{r} y^{-\nu} g_\nu(z)$, where the g_ν's are holomorphic functions on the upper half

plane with Fourier expansions $g_\nu(z) = \sum_{n=0}^{\infty} b_{\nu n} q^n$.

Example. The Eisenstein series $E_2 = 1 - 24q - \frac{3}{\pi y} + O(q^2)$ of weight 2 is a nearly holomorhic modular form.[3]

Note we have not yet shown that $y^m E_\lambda(z, m)$ satisfies (ii). It turns out that this indeed the case whenever $2 - \lambda \leq 2m \leq 0$ – this is an immediate consequence of equation 4, that we will see in a moment. We first put $\eta = z - \overline{z} = 2iy$ and define differential operators as follows:

$$\delta_k f := \frac{1}{2\pi i} \eta^{-k} \frac{\partial}{\partial z}(\eta^k f) = \frac{1}{2\pi i}\left(\frac{k}{\eta} + \frac{\partial}{\partial z}\right) f,$$

and inductively, $\delta_k^{(r+1)} := \delta_{k+2r} \delta_k^{(r)}$. Those definitions and several properties can be found in [Sh3]. A basic property is the fact that $\delta_k(f|_k\, \gamma) = (\delta_k f)|_{k+2}\, \gamma$, i.e. δ_k "increases" the weight of forms of weight k by 2. In particular, it preserves the space of nearly holomorphic modular forms.

We now give an identity showing how the delta operators interact with the non-holomorphic Eisenstein series. We have, for $r \in \mathbb{Z}_{\geq 0}$,

$$y^{-r} E_{\lambda+2r}(z, -r) = \frac{\Gamma(\lambda)}{\Gamma(\lambda + r)} (-4\pi)^r \delta_\lambda^{(r)} E_\lambda(z) \tag{4}$$

(this is a rearrangement of **9.4** of [Sh3]; it can be proven with a few lines of straightforward manipulations, which we have omitted for the sake of brevity). If $l + 2r < k$, 3 is holomorphic at $s = k - 1 - r$. Recalling the definition of the Petersson product and writing $\lambda = k - l - 2r$, we can substitute the above equation into 3 and rearrange to get (theorem 2 from [Sh1]):

$$D(k - 1 - r, f, g) = c\pi^k \langle f_\rho, g \cdot \delta_\lambda^{(r)} E_\lambda(z) \rangle, \tag{5}$$

where

$$c = \frac{\Gamma(k - l - 2r)}{\Gamma(k - 1 - r)\Gamma(k - l - r)} \cdot \frac{(-1)^r 4^{k-1}}{3}.$$

We now give a structure theorem for nearly holomorphic modular forms, shown in Lemma 7 of [Sh1]. The statement says roughly that most such forms can be expressed as

[3] It is a well known fact (see e.g. **Lemma 8.3.** of [Sh3]) that nearly holomorphic modular forms form a graded ring generated by E_2, E_4 and E_6. The proof given there is very similar to the proof of the structure theorem we will see in a moment.

linear combinations of usual modular forms to which we apply the δ_k operators. We first introduce some notation: write A_r for the set of all functions of the form $h(z) = \sum_{\nu=0}^{r} y^{-\nu} g_\nu(z)$, where the g_ν's are holomorphic functions with Fourier expansions, defined on the upper half plane. That is, elements of A_r are polynomials of degree r in y^{-1}, with the g_ν's as coefficients. In particular, any nearly holomorphic modular form is in A_r for some r, which we will call its degree of near holomorphy. The structure theorem is then as follows:

Theorem 3 (Structure of nearly holomorphic modular forms, lemma 7 from [Sh1])**.** *Let $k > 2r$. Any element h of A_r transforming like a modular form of weight k (i.e. any nearly holomorphic modular form of sufficiently large weight compared to its degree of holomorphy*[4]*) can be uniquely written as*

$$h(z) = \sum_{\nu=0}^{r} \delta_{k-2\nu}^{(\nu)} g_\nu,$$

with g_ν belonging to the space $G_{k-2\nu}$ of modular forms.

Proof. We use induction on r: the base case is clear, since A_0 is just the space of modular forms.

Now writing $y \circ \gamma(z)$ for the imaginary part of $\gamma(z)$, we have

$$y^{-1} \circ \gamma(z) = y^{-1} \cdot |cz+d|^2 = y^{-1} \cdot (cz+d)^2 - 2ci(cz+d)$$

(this can be easily shown with a few lines of manipulations). Take an element $h(z)$ in A_r transforming like a modular form. Write

$$h(z) = \sum_{\nu=0}^{r} y^{-\nu} g_\nu = h(\gamma(z))(cz+d)^{-k}.$$

We can now compare coefficients of y^{-r} and use our expression for $y^{-r} \circ \gamma(z) = (y^{-1} \circ \gamma(z))^r$ to get $(cz+d)^{2r-k} g_r(\gamma(z)) = g_r(z)$, so that g_r is a modular form of weight $k-2r$.

Note that $\delta_{k-2r}^{(r)} g_r$ is also in A_r and transforms like a form of weight k. We claim that if $k - 2r > 0$, the y^{-r} coefficient is a non-zero multiple of g_r. To see this, let h be any modular form of weight $l > 0$, and let h_n denote the y^{-n} coefficient of $\delta_l^{(n)} h$. Then $h_0 = h$, and it can be seen that h_{n+1} only depends on h_n. In fact, from the definition of the delta operators, and using the product rule and the fact that $\partial/\partial z = (1/2)(\partial/\partial x - i\partial/\partial y)$, one gets that $h_{n+1} = -\frac{l+n}{4\pi y} h_n$, which never vanishes, since $l > 0$.

This means that, for a suitable constant p, $h(z) - p\delta_{k-2r}^{(r)} g_r$ is in A_{r-1} and transforms like

[4] The proof shows *a posteriori* that any nearly holomorphic modular form has weight greater than or equal to twice its degree of holomorphy. Indeed, if $k - 2r < 0$ (or, incidentally, if $k - 2r = 2$), the form g_{k-2r}, i.e. the y^{-r} coefficient, must be 0.

a form of weight k, so the induction hypothesis applies (as $k > 2(r-1)$). Uniqueness follows by comparing coefficients. $\square$

2.4 Relating the values of the convolution to the Petersson norm

We now want to show that the ratio we get after dividing through by $\pi^k\langle f, f\rangle$ is in $K_f K_g$ (where K_h is the extension of the rationals by the Fourier coefficients of h). Because of 5, it suffices to prove that the ratio of the Petersson products

$$\frac{\langle f_\rho, g \cdot \delta_\lambda^{(r)} E_\lambda(z)\rangle}{\langle f, f\rangle}$$

lies in $K_f K_g$. The main step is to express $g \cdot \delta_\lambda^{(r)} E_\lambda(z)$ (which is a nearly holomorphic form of weight k) as a sum of nearly holomorphic forms which are either orthogonal to f, or a multiple of it. The structure theorem we have just seen does most of the job. Indeed, we have the following:

Lemma 4 (lemma 8 of [Sh1])**.** *Given a cusp form f of weight k and a modular form g of weight $l < k$ with $k = l + 2r$, $\langle f_\rho, \delta_\lambda^{(r)} g\rangle = 0$.*

Proof. The proof starts with a slightly more sophisticated version of the argument we used to find an integral expression for $D(s, f, g)$.

First note by inspection that $\delta_\lambda^{(r)} g = (2\pi y)^{-r} \sum_{\nu=0}^{r} c_\nu (2iy)^{\nu-r} (\partial/\partial z)^\nu g$ for some coefficients $c_\nu \in \mathbb{Z}$. Writing $g = \sum_{n=0}^{\infty} b_n q^n$, we have $(\partial/\partial z)^\nu g = (2\pi i)^\nu \sum_{n=0}^{\infty} b_n n^\nu q^n$, so that

$$\delta_\lambda^{(r)} g = \sum_{\nu=0}^{r} (-4\pi y)^{\nu-r} c_\nu \sum_{n=0}^{\infty} b_n n^\nu q^n.$$

Now

$$\Gamma(s+\nu-r) = \int_0^\infty u^{s+\nu-r-1} e^{-u}\, du = (4\pi n)^{s+\nu-r} \int_0^\infty y^{s+\nu-r-1} e^{-4\pi n y}\, dy,$$

giving

$$\begin{aligned}(-1)^{\nu-r} c_v (4\pi)^{-s} D(s-r, f, g)\Gamma(s+\nu-r) &= c_v(-4\pi)^{\nu-r} \sum_{n=0}^{\infty} a_n b_n n^\nu \int_0^\infty y^{s+\nu-r-1} e^{-4ny}\, dy \\ &= \int_0^\infty y^{s-1} \sum_{n=0}^{\infty} a_n (-4\pi y)^{\nu-r} c_\nu b_n n^\nu\, dy.\end{aligned} \tag{6}$$

Summing up over ν, the coefficients of $\delta_\lambda^{(r)}$ appear inside the sum over n, and we apply Parseval's theorem to get that the above is equal to

$$\int_0^\infty y^{s-1} \int_0^1 \overline{f_\rho} \delta_\lambda^{(r)} g \, dx \, dy.$$

Use the same unfolding argument as before to rewrite the above integral and get

$$(4\pi)^{-s} D(s-r, f, g) \sum_{\nu=0}^{r} (-1)^{\nu-r} c_v \Gamma(s+\nu-r) = \int_\Phi \overline{f_\rho} \delta_\lambda^{(r)} g \cdot E_0(z, s+1-k) y^{s-1} \, dx \, dy.$$

At this point, we take the residue at $s = k$ of both sides.

We claim that on the right hand side, we get a non-zero constant times $\langle f_\rho, \delta_\lambda^{(r)} g \rangle$. Rankin shows, as part of Theorem 3 in [Ran], a similar statement whose proof can be adapted to our situation. The whole proof is not particularly difficult, however it is long and messy, so we only give a very brief outline: Rankin shows (in a different notation) that $y^{s+1-k} E_0(z, s+1-k)$ times a few factors holomorphic and non-vanishing at $s = k$ can be written as an entire function of s plus $1/(s-k)(s+1-k)$. We can thus split our integral above into two. Rankin then explains that the first integral (involving the entire function) is still an entire function of s since it converges absolutely, while for the other one, we can take $1/(s-k)(s+1-k)$ outside (since it doesn't depend on x, y), and we're left with the Petersson product we're interested in times $1/(s-k)$ times a few functions holomorphic and non vanishing at k, and our claim immediately follows.

On the left hand side, we have a holomorphic function in s (this is stated as a well known fact in [Sh1]), so the residue is 0, and we are done.

□

We can thus use the structure theorem to write

$$g \cdot \delta_\lambda^{(r)} E_\lambda(z) = \sum_{\nu=0}^{r'} \delta_{k-2\nu}^{(\nu)} g_\nu,$$

where r' is $r+1$ if $\lambda = 2$ and r otherwise (this is because the Eisenstein series of weight 2 has a y^{-1} term, so belongs to A_1). By the lemma we have just shown, all of the terms with $\nu > 0$ are orthogonal to f, so that

$$\langle f_\rho, g \cdot \delta_\lambda^{(r)} E_\lambda(z) \rangle = \langle f_\rho, g_0 \rangle;$$

it can be easily checked that $K_{g_0} = K_g$. For a normalised eigenform f and an element h in M_k, the ratio $\langle f_\rho, h \rangle / \langle f, f \rangle$ belongs to $K_{f_\rho} K_h = K_f K_h$, and so we are done.

2.5 A worked example

Let

$$F = q - 528q^2 + O(q^3) \quad \text{and} \quad G = q - 24q^2 + O(q^3)$$

be the normalised cusp forms of weights 18 and 12 respectively for the full modular group. Let

$$Z(s) = \frac{L(F \otimes G, s)(s-1)!\,(s-12)!}{3^{-1} \cdot \pi^{2s-10} \cdot 2^{2s} \cdot \langle F, F \rangle}.$$

We will use the theory described so far to find the values $Z(s)$ for $s = 12$, 13, 14, 15, 16, 17. My supervisor has computed those values numerically to 250 decimal places, and they agree with the values we get here.

First note that $Z(s) = Z(29-s)$[5]. This means that we only need to compute the values of Z at 15, 16 and 17. Our starting point is Theorem 2 from [Sh1] (equation 5 here). The statement becomes the following after substituting in the weights:

$$L(F \otimes G, 17-r) = \zeta(6-2r) c \pi^{18} \langle F, G \cdot \delta_\lambda^{(r)} E_\lambda(z) \rangle,$$

where $\lambda = 6 - 2r$, and

$$c = \frac{\Gamma(6-2r)}{\Gamma(17-r)\Gamma(6-r)} \cdot \frac{(-1)^r 4^{17}}{3}.$$

We can rearrange this and divide through by $\langle F, F \rangle$ to get :

$$\frac{L(F \otimes G, 17-r)(17-r-1)!\,(17-r-12)!}{3^{-1} \cdot \pi^{2(17-r)-10} \cdot 2^{2(17-r)} \cdot \langle F, F \rangle} = \zeta(6-2r)\Gamma(6-2r)(-4)^r \pi^{2r-6} \cdot \frac{\langle F, G \cdot \delta_\lambda^{(r)} E_\lambda \rangle}{\langle F, F \rangle}$$

Note that for $r = 0, 1, 2$, the left hand side gives us exactly $Z(17)$, $Z(16)$ and $Z(15)$ respectively, so we just need to compute the right hand side to get our values. For the value on the edge of the critical strip, i.e. $r = 0$, this is easy:

$$Z(17) = \zeta(6)\Gamma(6)\pi^{-6} \cdot \frac{\langle F, G \cdot \delta_6^{(0)} E_6 \rangle}{\langle F, F \rangle},$$

and since $\delta_6^{(0)}$ is the identity, and $G \cdot E_6 = F$, the ratio of the Petersson products is 1, giving

$$Z(12) = Z(17) = \frac{\pi^6}{945} \cdot 5! \cdot \pi^{-6} = \frac{8}{63}.$$

For the values deeper inside the critical strip, we need to do a bit more work. In order to compute the ratios of the Petersson products, we will work in the space of nearly holomorphic modular forms of weight 18, where both F and $G \cdot \delta_\lambda^{(r)} E_\lambda(z)$ lie. Using our

[5] This follows from equation (22) in [Sh2] (the notation is different) after a bit of rearranging.

structure theorem, we want an expression

$$G \cdot \delta_\lambda^{(r)} E_\lambda(z) = \sum_{\nu=0}^{r'} \delta_{18-2\nu}^{(\nu)} g_\nu, \tag{7}$$

where the g_ν's are modular forms of weight $18 - 2\nu$, and r' is 1 for the outermost value, and 3 for the innermost one (since $\delta_4 E_4(z)$ is in A_1, and $\delta_\lambda^{(2)} E_\lambda(2)$ is in A_3). Then $\langle F, \delta_{18-2\nu}^{(\nu)} g_\nu \rangle = 0$ for all $\nu > 0$. This means that the product only depends on the holomorphic part of $G \cdot \delta_\lambda^{(r)} E_\lambda(z)$, which is a modular (in fact, cusp) form of weight 18, i.e. a multiple of F.

Note that in practice, for modular forms of weight k, we only need to deal with the first $\lfloor k/12 \rfloor + 1$ terms of the q-expansions, since those determine modular forms uniquely. We have:

$$E_2 = 1 - 24q - \frac{3}{\pi y} + O(q^2) \quad \text{and} \quad E_4 = 1 + 240q + O(q^2),$$

and we can compute

$$\begin{aligned} \delta_4 E_4 &= \frac{1}{2\pi i} \left(\frac{\partial}{\partial z} + \frac{4}{2iy} \right) E_4 \\ &= 240q - \frac{1}{\pi y} (1 + 240q) + O(q^2) \end{aligned}$$

and

$$\begin{aligned} \delta_2^{(2)} E_2 &= \delta_4 \frac{1}{2\pi i} \left(\frac{\partial}{\partial z} + \frac{2}{2iy} \right) E_2 \\ &= \frac{1}{2\pi i} \left(\frac{\partial}{\partial z} + \frac{4}{2iy} \right) \left(-24q - \frac{1}{2\pi y} (1 - 24q) + \frac{3}{4\pi^2 y^2} + O(q^2) \right) \\ &= -24q + \frac{36q}{\pi y} + \frac{3}{8\pi^2 y^2} (1 - 24q) - \frac{3}{8\pi^3 y^3} + O(q^2), \end{aligned}$$

where for the last equality, we have used that $\frac{\partial}{\partial z}(y^{-1}) = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(y^{-1}) = \frac{iy^{-2}}{2}$, and the chain rule. We thus have

$$G \cdot \delta_4 E_4(z) = -\frac{q}{\pi y} + O(q^2)$$

and

$$G \cdot \delta_2^{(2)} E_2 = \frac{3q}{8\pi^2 y^2} - \frac{3q}{8\pi^3 y^3} + O(q^2).$$

Now we need to find expressions like in 7 for those two nearly holomorphic forms. In general, this can be done by taking a basis of the spaces of modular forms of smaller weights, applying the differential operators to it, then equating coefficients. M_{18} has basis $\{E_{18}, F\}$, M_{16} has basis $\{E_{16}, f_{16}\}$ where f_{16} is the normalised cusp form of weight 16, M_{14} has basis $\{E_{14}\}$ and M_{12} has basis $\{E_{12}, G\}$.

We have

$$\delta_{16} f_{16} = q - \frac{4q}{\pi y} + O(q^2).$$

We note that

$$G \cdot \delta_4 E_4(z) = \frac{1}{4}\delta_{16} f_{16} - \frac{1}{4}F,$$

whence the ratio of the Petersson products is -1/4, and we can compute

$$Z(13) = Z(16) = \frac{\pi^4}{90} \cdot 3! \cdot (-4) \cdot \pi^{-4} \cdot \left(-\frac{1}{4}\right) = \frac{1}{15}.$$

Similarly, a lengthier calculation gives us

$$\delta_{12}^{(3)} G = q - \frac{21q}{2\pi y} + \frac{273q}{8\pi^2 y^2} - \frac{273q}{8\pi^3 y^3} + O(q^2),$$

and we note that

$$G\delta_2^{(2)} E_2 = \frac{1}{91}\delta_{12}^{(3)} G - \frac{3}{104}\delta_{16} f_{16} + \frac{1}{56}F,$$

giving

$$Z(14) = Z(15) = \frac{\pi^2}{6} \cdot 1! \cdot (-4)^2 \cdot \pi^{-2} \cdot \frac{1}{56} = \frac{1}{21}.$$

3 The case $f = g$

In this section, we will investigate some special values of the convolution $L(f \otimes f, s)$. The exact method we have used in the previous section does not work here, because the non-holomorphic Eisenstein series $E_0(z, s)$ defined in 2 is not a nearly holomorphic modular form. We need a few new ideas to tackle this problem. As before, we let $f = \sum\limits_{n=1}^{\infty} a_n q^n$ be a primitive cusp form of weight k, and

$$L(f \otimes f, s) = \zeta(2s - 2k + 2) \sum_{n=1}^{\infty} {a_n}^2 n^{-s}.$$

Sturm (in his 1977 Princeton thesis) and Zagier in [Zag] independently discovered identities for special values of the convolution in terms of the Peterson norm of f. We will follow the method described by Zagier in §5 of his paper to obtain these identities.

3.1 The general idea

In Zagier's notation, we put $D_f(s) = \zeta(s-k+1)^{-1}L(f\otimes f, s)$. In order to be consistent with his notation, we will ignore the factor involving the size of the fundamental domain when defining the Petersson product, i.e.

$$\langle f, g\rangle = \int_{\Phi} \overline{f(z)} g(z) y^{k-2}\, dx\, dy.$$

The result we will look at is Theorem 2 of [Zag], which relates the values of $D_f(s)$ at $k+r-1$, where $r = 1, 3, ..., k-3$,[6] to the Petersson norm of f via a similar expression to the one we have seen in the previous section.

The method described by Zagier shows a few conceptual similarities to Shimura's, though the details are different: we want to obtain an expression of $D_f(s)$ ($D(s, f, g)$ in the previous part) in terms of the product of f with something, then compute the ratio of that product to the norm of f. In the previous part, that something was a nearly holomorphic modular form, and a lot of the work went in computing the ratio; this time, it is a modular form $C_{k,r}$, and computing the ratio is easy — most of the work goes in relating it to $D_f(r)$. To do this, we define a bilinear differential operator F_ν introduced by Cohen, which allows us to express $C_{k,r}$ as the derivative of a pair of half integral weight modular forms. Finally, we look at how the Petersson product interacts with this operator, and that gives us what we need to relate $\langle f, C_{k,r}\rangle$ to $D_f(r)$.

3.2 Modular forms of half integral weight

We start with some general definitions given in [Zag], that we will use throughout the section. Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup, $k > 0$ a real number. We consider functions $v : \Gamma \mapsto \{t \in \mathbb{C} : |t| = 1\}$ ("multiplier systems") such that the automorphy factor

$$J(\gamma, z) = v(\gamma)^{-1}(cz+d)^{-k} \qquad \left(\text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ and } z \in \mathbb{H}\right)$$

satisfies the cocycle property

$$J(\gamma_1\gamma_2, z) = J(\gamma_1, \gamma_2 z) J(\gamma_2, z)$$

[6] Zagier shows the result for $r = k-1$ as well, but that case presents a few extra convergence issues and no fundamentally new ideas, so we will not worry about it here.

and such that $v(\gamma) = 1$ for $\gamma \in \Gamma_\infty = \Gamma \cap SL_2(\mathbb{Z})_\infty$. Write $M_k(\Gamma, v)$, $S_k(\Gamma, v)$ for the spaces of modular (respectively cusp) forms on Γ which transform by $f(z) = J(\gamma, z)f(\gamma z)$. Thus the product of forms in $M_{k_1}(\Gamma, v_1)$ and $M_{k_2}(\Gamma, v_2)$ lies in $M_{k_1+k_2}(\Gamma, v_1v_2)$.

For $k > 2$, define the Eisenstein series

$$E'_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} J(\gamma, z) \in M_k(\Gamma, v),$$

and more generally for $n \in \mathbb{N}$, the Poincaré series

$$G'_n(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} J(\gamma, z) e^{2\pi i n \gamma z/w},$$

where w is the index of Γ_∞ in $SL_2(\mathbb{Z})_\infty$. It is a well known fact[7] (in the case where $k \in \mathbb{Z}$, but it is easy to generalise) that for $n \geq 1$, G'_n is a cusp form of weight k, and for a form $h = \sum_{n=0}^{\infty} c_n q^n$ in $M_k(\Gamma, v)$,

$$\langle h, G'_n \rangle = \frac{\Gamma(k-1) w^k}{(4\pi n)^{k-1}} \; c_n. \tag{8}$$

We are now ready to talk about half integral weight modular forms. We introduce them through an example, Jacobi's theta function

$$\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}.$$

One can ask whether θ has any nice transformation properties with respect to matrices in $SL_2(\mathbb{Z})$. As it turns out, this is not the case for the full modular group. However, we can write an expression for how θ transforms with the action of $\Gamma_0(4)$. Deriving the whole expression is a bit messy, so we only give a proof for a special case and then state the general result.

We start by showing that

$$\theta\left(\frac{-1}{4z}\right) = \sqrt{\frac{2z}{i}}\theta(z)$$

(we always choose the square root with argument greater than $-\pi/2$ and less than or equal to $\pi/2$).

[7] See for example p. 37 of Gunning's *Lecture notes on modular forms*, 1962. The proof of the formula for the inner product given there uses an unfolding argument similar to what we did in the first part.

Recall that the Fourier transform of f (where f decreases sufficiently fast, e.g. exponentially) is given by

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixt}\, dx.$$

Now note that the function $e^{-\pi t^2}$ is its own Fourier transform. Indeed, its Fourier transform is

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi ixt}\, dx &= \int_{-\infty}^{\infty} e^{-\pi(x^2-2ixt)}\, dx \\ &= \int_{-\infty}^{\infty} e^{-\pi(x-it)^2-\pi t^2}\, dx \\ &= e^{-\pi t^2} \int_{-\infty}^{\infty} e^{-\pi(x-it)^2}\, dx,\end{aligned}$$

and it can be shown using standard complex analysis techniques (looking at a contour integral over a rectangle, using Cauchy's theorem and taking a limit) and the formula $\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$, that the above integral is always 1.

Putting $f_a(t) = e^{-\pi at^2}$ (so that $f_1(t) = e^{-\pi t^2}$ and $f_a(t) = f_1(\sqrt{a}t)$), we can do a change of variables $y = \sqrt{a}x$ to get

$$\begin{aligned}\hat{f}_a(t) &= \int_{-\infty}^{\infty} f_a(x)e^{-2\pi ixt}\, dx \\ &= \int_{-\infty}^{\infty} f_1(\sqrt{a}x)e^{-2\pi ixt}\, dx \\ &= \int_{-\infty}^{\infty} f_1(y)e^{-2\pi iy\frac{t}{\sqrt{a}}} = \frac{1}{\sqrt{a}}\, f_1\left(\frac{t}{\sqrt{a}}\right).\end{aligned}$$

Now we set $a = \frac{i}{2z}$, and apply the Poisson summation formula[8] to f_a to get

$$\theta\left(\frac{-1}{4z}\right) = \sum_{n\in\mathbb{Z}} e^{\frac{-2\pi i}{4z}n^2} = \sqrt{\frac{2z}{i}} \sum_{n\in\mathbb{Z}} e^{2\pi izn^2} = \sqrt{\frac{2z}{i}}\theta(z).$$

Using this expression, we can describe how θ transforms under the action of $\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}$. We

[8]This is a standard result of Fourier analysis saying that, for a sufficiently nice function f, $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$.

have[9]

$$\begin{aligned}\theta\left(\frac{z}{4z+1}\right) &= \theta\left(\frac{-1}{4\left(-\frac{1}{4z}-1\right)}\right) \\ &= \sqrt{\frac{2}{i}\left(-\frac{1}{4z}-1\right)}\;\theta\left(-\frac{1}{4z}-1\right) \qquad \text{by the previous part} \\ &= \sqrt{\frac{4z+1}{2z/i}}\;\theta\left(-\frac{1}{4z}\right) \qquad \text{since } \theta(z)=\theta(z+1) \\ &= \epsilon(4z+1)^{\frac{1}{2}}\;\theta(z) \qquad \text{by previous part,}\end{aligned}$$

where ϵ is a number with $\epsilon^2 = 1$, which appears due to the fact that the product of two roots is only the root of the product up to a factor of ± 1 (otherwise we could potentially get outrageous results, like $-1 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$). For our transformation, $\epsilon = 1$.

Now $\left(\begin{smallmatrix}1 & 0\\ 4 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ generate $\Gamma_0(4)$. Assuming that under two matrices γ_1 and γ_2, θ transforms via $\theta(\gamma_i z) = \epsilon_{\gamma_i}(c_i z + d_i)^{\frac{1}{2}}\theta(z)$ where ϵ_{γ_i} is a complex number of modulus 1, it will transform under $\gamma_1\gamma_2$ via a similar expression (with an $\epsilon_{\gamma_1\gamma_2}$ that is not simply the product of the other two – see our above explanation). It turns out that in general,[10] for a matrix γ in $\Gamma_0(4)$ with entries a, b, c, d, we have

$$\theta(\gamma z) = \epsilon_\gamma(cz+d)^{\frac{1}{2}}\theta(z),$$

where $\epsilon_\gamma = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-\frac{1}{2}}$ is expressed in terms of the Kronecker symbol $\left(\frac{a}{n}\right)$, which is an extension of the Legendre quadratic residue symbol to all integers.

We will call θ a modular form on $\Gamma_0(4)$ with weight $\frac{1}{2}$ and trivial character. In general, we define an automorphy factor on $\Gamma_0(4)$ by $j(\gamma, z) = \theta(z)/\theta(\gamma z) = \epsilon_\gamma^{-1}(cz+d)^{-\frac{1}{2}}$, with multiplier system ϵ_γ. Then for $r \in \mathbb{N}$, we define modular forms of weight $r + \frac{1}{2}$ with character χ as holomorphic (and holomorphic at the cusps) functions h on the upper half plane which transform via $h(z) = \chi(d) j(\gamma, z)^{2r+1} f(\gamma z)$.

Cohen now defines, for r a natural number, functions

$$\mathcal{H}_r = \sum_{N=0}^{\infty} H(r, N) q^N.$$

The coefficients $H(r, N)$ have a fairly complicated definition involving values of L-functions of quadratic characters, but we will not go into that. For our purposes, it suffices to know

[9] Adapted from chapter 3 of Zagier's 1991 lecture notes, *Modular Forms of One Variable*.

[10] See e.g. §2 of [Sh4].

that they are rational numbers with bounded denominators, and that Cohen has computed a table with some of the values. He also shows that[11]

$$\mathcal{H}_r = \zeta(1-2r)\left(E^{(4)}_{r+\frac{1}{2}}(z) + (1-i)(4z)^{-r-\frac{1}{2}}E^{(4)}_{r+\frac{1}{2}}\left(\frac{-1}{4z}\right)\right), \tag{9}$$

where

$$E^{(4)}_{r+\frac{1}{2}}(z) = \sum_{\gamma\in\Gamma_0(4)_\infty\backslash\Gamma_0(4)} \frac{\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}}{(cz+d)^{r+1/2}}.$$

In particular, $\mathcal{H}_r$ is a nearly holomorphic modular form of weight $r+1/2$.

3.3 The F_ν operators

Cohen defines in [Coh] bilinear operators involving derivatives of smooth functions: for real numbers k_1, k_2, set

$$F_\nu(f_1, f_2)(z) = \sum_{\mu=0}^{\nu}(-1)^{\nu-\mu}\binom{\nu}{\mu}\frac{\Gamma(k_1+\nu)\Gamma(k_2+\nu)}{\Gamma(k_1+\mu)\Gamma(k_2+\nu-\mu)}\ \frac{\partial^\nu f_1}{\partial z^\mu}\ \frac{\partial^{\nu-\mu} f_2}{\partial z^{\nu-\mu}}.$$

Those operators satisfy the following property for $\gamma \in GL_2^+(\mathbb{R})$, shown by Cohen in theorem 7.1 of his paper: $F_\nu(f_1\,|_{k_1}\gamma, f_2\,|_{k_2}\gamma) = F_\nu(f_1, f_2)\,|_{k_1+k_2+2\nu}\gamma$ (where $(f|_k\gamma)(z) = (cz+d)^{-k}f(\gamma z)$). In particular, if f_1 and f_2 are modular forms with weights k_1 and k_2 and multiplier systems v_1, v_2, then $F_\nu(f_1, f_2)$ is a modular form of weight $k_1+k_2+2\nu$ with multiplier system v_1v_2.

Proof. ([Coh] Theorem 7.1 a))

First put $E(z) = (z-\bar{z})^{-1}$. It is easy to check that $E|_2\gamma = E - c(cz+d)^{-1}$ (this is very similar to one of the equalities we used in the proof of the structure theorem for nearly holomorphic modular forms), and that $\partial_z E = -E^2$ (where ∂_z means $\partial/\partial z$). With this, we have, for h holomorphic and $\kappa \in \mathbb{R}$:

$$\begin{aligned}((\partial_z h + \kappa E h)|_{\kappa+2}\gamma)(z) &= (cz+d)^{-\kappa-2}(\partial_z h)(\gamma z) + (cz+d)^{-\kappa-2}\kappa E(\gamma z)h(\gamma z)\\ &= (cz+d)^{-\kappa-2}(\partial_z h)(\gamma z) + (cz+d)^{-\kappa}h(\gamma z)\kappa(cz+d)^{-2}E(\gamma z)\\ &= (cz+d)^{-\kappa-2}(\partial_z h)(\gamma z) - c\kappa(cz+d)^{-\kappa-1}h(\gamma z) + \kappa E(z)(cz+d)^{-\kappa}h(\gamma z)\\ &= \partial_z\left((h|_\kappa\gamma)(z)\right) + \kappa E(h|_\kappa\gamma)(z)\end{aligned}$$

[11] We have taken the following two formulae from Zagier's paper. The definition given in Cohen's paper for the Eisenstein series is slightly different. Somewhat confusingly, the definition we have given does not seem to agree with the conventions for Eisenstein series Zagier uses later in his paper – this series appears to be a twist of what it should be with said conventions.

Now set

$$F_r(h) = \sum_{l=0}^{r} \binom{r}{l} \frac{(\kappa + r - 1)!}{(\kappa + l - 1)!} E^{r-l} \partial_z^l h.$$

Through a manipulation very similar to the one we will use in a moment to prove equation 11, one shows that $F_{r+1}(h) = \partial_z(F_r(h)) + (\kappa + 2r)EF_r(h)$. With this, one can now show by induction that $F_r(h)|_{\kappa+2r}\gamma = F_r(h|_\kappa\gamma)$. The base case $r = 0$ is true, since F_0 is just the identity. Assuming it holds for some $r \geq 0$, we have:

$$\begin{aligned} F_{r+1}(h)|_{\kappa+2r+2}\gamma &= (\partial_z(F_r(h)) + (\kappa + 2r)EF_r(h))|_{\kappa+2r+2}\gamma \\ &= \partial_z\left(F_r(h)|_{\kappa+2r}\gamma\right) + (\kappa + 2r)E(F_r(h)|_{\kappa+2r}\gamma) \\ &= \partial_z F_r(h|_\kappa\gamma) + (\kappa + 2r)EF_r(h|_\kappa\gamma) \\ &= F_{r+1}(h|_\kappa\gamma), \end{aligned}$$

where the first and last equalities come from the recursive formula for F_r, the second one is from the identity above (where we replace h with $F_r(h)$ and κ with $\kappa + 2r$), and the third one is the induction hypothesis.

Put

$$G_h(z,t) = \sum_{r \geq 0} \frac{t^{2r} F_r(h)}{r!\,(\kappa + r - 1)!}.$$

It is straightforward to check that

$$G_h(z,t) = e^{t^2 E} \sum_{l \geq 0} \frac{t^{2l}}{l!\,(\kappa + l - 1)!} \partial_z^l h$$

(just write the Taylor expansion for $e^{t^2 E}$ and equate coefficients of the product of the two series). Again, equating coefficients, one gets that

$$G_{f_1}(z,t)G_{f_2}(z,it) = \sum_{\nu \geq 0} \frac{t^{2\nu} F_\nu(f_1, f_2)}{\nu!\,(k_1 + \nu - 1)!\,(k_2 + \nu - 1)!}.$$

Now note that, from our earlier identity for F, one gets that

$$G_h\left(\gamma z, \frac{t}{cz + d}\right) = (cz + d)^\kappa G_{(h|_\kappa\gamma)}(z,t).$$

Putting all of the above together, we have

$$\sum_{\nu\geq 0}\frac{t^{2\nu}F_\nu(f_1,f_2)(\gamma z)(cz+d)^{-2n}}{\nu!\,(k_1+\nu-1)!\,(k_2+\nu-1)!}=G_{f_1}\left(\gamma z,\frac{t}{cz+d}\right)\cdot G_{f_2}\left(\gamma z,\frac{it}{cz+d}\right)$$
$$=(cz+d)^{k_1+k_2}G_{(f_1|_{k_1}\gamma)}(z,t)G_{(f_2|_{k_2}\gamma)}(z,it)$$
$$=\sum_{\nu\geq 0}\frac{t^{2\nu}(cz+d)^{k_1+k_2}F_\nu(f_1|_{k_1}\gamma,f_2|_{k_2}\gamma)(z)}{\nu!\,(k_1+\nu-1)!\,(k_2+\nu-1)!},$$

and comparing coefficients (in t) gives us the desired result. □

We now give an expression for the product of f with the Cohen operator applied to a modular form and an Eisenstein series (Proposition 6 of [Coh]):

Proposition 5. *Let $J_i(\gamma,z)=v_i(\gamma)^{-1}(cz+d)^{-k_i}$, $i=1,2$ be two automorphy factors on Γ', where k_1,k_2 are real numbers with $k_s\geq k_1+2>2$. Let ν be a non-negative integer, and put $k=k_1+k_2+2\nu$, $v=v_1v_2$. Let $f(z)=\sum_{n=1}^{\infty}a_ne^{2\pi inz/w}$ and $g(z)=\sum_{n=0}^{\infty}b_ne^{2\pi inz/w}$ be modular forms in $S_k(\Gamma',v)$ and $M_{k_1}(\Gamma',v_1)$ respectively, and E'_{k_2} the Eisenstein series in $M_{k_2}(\Gamma',v_2)$. Then the Petersson product of f and $F_\nu(g,E'_{k_2})$ is given by*

$$\langle f,F_\nu(g,E'_{k_2})\rangle=(2\pi i)^\nu\frac{\Gamma(k-1)}{(4\pi)^{k-1}\Gamma(k_2)}w^{k-\nu}\sum_{n=1}^{\infty}\frac{a_n\overline{b_n}}{n^{k_1+k_2+\nu-1}}.\tag{10}$$

Proof. We begin by showing by induction that

$$g^{(\nu)}(\gamma z)=v_1(\gamma)\sum_{\mu=0}^{\nu}\binom{\nu}{\mu}\frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)}c^{\nu-\mu}(cz+d)^{k_1+\nu+\mu}g^{(\mu)}(z).\tag{11}$$

The base case is just the transformation law for g, so let us assume that the statement holds for ν. We then have:

$$
\begin{aligned}
g^{(\nu+1)}(\gamma z) &= (cz+d)^2 \frac{d}{dz} g^{(\nu)}(\gamma z) \qquad \text{(by chain rule)} \\
&= (cz+d)^2 \frac{d}{dz} v_1(\gamma) \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)} c^{\nu-\mu} (cz+d)^{k_1+\nu+\mu} g^{(\mu)}(z) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by induction hypothesis)} \\
&= v_1(\gamma) \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)} c^{\nu-\mu+1} (k_1+\nu+\mu)(cz+d)^{k_1+\nu+\mu+1} g^{(\mu)}(z) \\
&\qquad + v_1(\gamma) \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)} c^{\nu-\mu} (cz+d)^{k_1+\nu+\mu+2} g^{(\mu+1)}(z) \\
&\qquad\qquad\qquad\qquad \text{(by linearity of d/dz and product rule)} \\
&= v_1(\gamma) \frac{\Gamma(k_1+\nu)}{\Gamma(k_1)} c^{\nu}(k_1+\nu)(cz+d)^{k_1+\nu+1} g(z) + v_1(\gamma)(cz+d)^{k_1+2\nu+2} g^{(\nu+1)}(z) \\
&\qquad + v_1(\gamma) \sum_{\mu=1}^{\nu} c^{\nu-\mu+1}(cz+d)^{k_1+\nu+\mu+1} \\
&\qquad\qquad \cdot \left(\binom{\nu}{\mu} \frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)} (k_1+\nu+\mu) + \binom{\nu}{\mu-1} \frac{\Gamma(k_1+\nu+1)}{\Gamma(k_1+\mu)} \right) \\
&\text{(by reindexing the 2nd sum, then writting the } \mu = 0 \text{ and } \mu = \nu+1 \text{ terms separately)} \\
&= v_1(\gamma) \sum_{\mu=0}^{\nu+1} \binom{\nu+1}{\mu} \frac{\Gamma(k_1+\nu+1)}{\Gamma(k_1+\mu)} c^{\nu+1-\mu} (cz+d)^{k_1+\nu+1+\mu} g^{(\mu)}(z)
\end{aligned}
$$

as required, where in the last line we have rewritten the expression in brackets by using $\Gamma(z+1) = z\Gamma(z)$ and $\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$, and grouped the terms together.

Now $g^{(\nu)}$ has Fourier expansion

$$
g^{(\nu)}(z) = \left(\frac{2\pi i}{w} \right)^{\nu} \sum_{n=0}^{\infty} n^{\nu} b_n q^{n/w}.
$$

From the definition of the Poincaré series $G'_n(z)$, and using 11, we get

$$\begin{aligned}
\left(\frac{2\pi i}{w}\right)^{\nu}\sum_{n=0}^{\infty} n^{\nu} b_n G'_n(z) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} v(\gamma)^{-1}(cz+d)^{-k} g^{(\nu)}(\gamma z) \\
&= \sum_{\mu=0}^{\nu}\binom{\nu}{\mu}\frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)} g^{(\mu)}(z) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} v_2(\gamma)^{-1} c^{\nu-\mu}(cz+d)^{-k_2-\nu+\mu} \\
&= \sum_{\mu=0}^{\nu}\binom{\nu}{\mu}\frac{\Gamma(k_1+\nu)}{\Gamma(k_1+\mu)} g^{(\mu)}(z)\cdot(-1)^{\nu-\mu}\frac{\Gamma(k_2)}{\Gamma(k_2+\nu-\mu)} E_{k_2}^{\prime(\nu-\mu)}(z) \\
&= \frac{\Gamma(k_2)}{\Gamma(k_2+\nu)} F_\nu(g, E'_{k_2}).
\end{aligned}$$

Using 8, we get our result by taking the inner product of f with each side.

□

We are now ready to apply what we know about Cohen's operators to get a transparent expression for $D_f(r)$ (at our special values) in terms of the norm of f.

3.4 Relating the values of $D_f(s)$ to the Petersson norm

We start with the following definition: put

$$C_{k,r}(z) = \sum_{m=0}^{\infty}\left(\sum_{\substack{t\in\mathbb{Z}\\ t^2\le 4m}} P_{k,r}(t,m) H(r, 4m-t^2)\right) q^m,$$

where the $H(r,N)$ are the coefficients of Cohen's $\mathcal{H}_r$ function, and $P_{k,r}(t,m)$ is defined as the coefficient of x^{k-r-1} in the expansion of $(1-tx+mx^2)^{-r}$.

It turns out that $C_{k,r}$ satisfies the following identity:

$$C_{k,r} = (2\pi i)^{-\nu}\frac{\Gamma(\nu+r)}{\Gamma(r)\Gamma(k-r)} F_\nu(\theta, \mathcal{H}_r)\,|U_4, \tag{12}$$

with $\nu = (k-r-1)/2$, and where U_4 is the operator sending $\sum a_n q^n$ to $\sum a_{4n} q^n$. We only give a loose outline of the proof, most of which can be found in [Coh].

Outline of proof. Cohen first gives an alternative formula for $F_\nu(f_1, f_2)$ in Theorem 7.1 b) of his paper. Using it, he then computes the coefficients of $F_\nu(\theta, \mathcal{H}_r)$. He relates them to the coefficients of $(1-tx+mx^2)^{-r}$ by computing an expansion for this expression in Lemma 7.5 (he does so by writing a differential equation for it, thus obtaining a recurrence relation of degree 2 for the coefficients of a candidate expansion, then shows that the expansion he gives satisfies that relation). He expresses the result at the end of the proof of Theorem 6.1 (he writes it as an equality of power series whose coefficients are modular forms of level 4; we are interested in the $k-1-r$'th coefficient).

At this stage, we are still at level 4. To finish, Cohen passes to level 1 in two steps, using two lemmas from a paper by Li (the first one states that U_2 will map the function we are interested in to level 2 because 2^2 divides 4, and the second one states that applying U_2 again will send our new function from level 2 to level 1 because every odd coefficient vanishes).

This identity immediately shows that $C_{k,r}$ is a cusp form in S_k. It remains to compute the Petersson product of f with it. The computation is messy, and consists of using identity 12, then substituting 9 inside it and using 10 to finally express the product as a bunch of factors times

$$\sum_{n=1}^{\infty} \frac{a_n \bar{b_n}}{n^{k_1+k_2+\nu-1}},$$

where the b_n's are the coefficients of the θ function. Since those coefficients are 0 for non-squares and 2 for squares, the above can be written as

$$2\sum_{n=1}^{\infty} \frac{a_{n^2}}{n^{2(k_1+k_2+\nu-1)}} = 2\sum_{n=1}^{\infty} \frac{a_{n^2}}{n^{r+k-1}}.$$

In fact, one can check, using the multiplicative properties of the a_i's, that $D_f(s)$ satisfies

$$D_f(s) = \zeta(2s-2k+2)\sum_{n=1}^{\infty} a_{n^2} n^{-s},$$

so what we have above is actually a multiple of $D_f(r+k-1)$.

We end up with our main identity (theorem 2 of [Zag]):

$$\langle f, C_{k,r}\rangle = -\frac{(r+k-2)!\,(k-2)!}{(k-r-1)!} \cdot \frac{1}{4^{r+k-2}\pi^{2r+k-1}} D_f(r+k-1)$$

for $r = 3, 5, ..., k-3$.

3.5 A worked example

Like in the last section, let $F = q - 528q^2 + O(q^3)$ be the cusp form of weight 18. We will compute the ratio

$$\frac{L(F \otimes F, 20)}{\pi^{23}\langle F, F\rangle \zeta(3)} = \frac{D_f(20)}{\pi^{23}\langle F, F\rangle}.$$

Putting $k = 18$, $r = 3$ in our main identity, we get

$$\langle f, C_{18,3}\rangle = -\frac{19!\,16!}{14!} \cdot \frac{1}{4^{19}\pi^{23}} D_f(20),$$

and by rearranging, we get that our ratio is

$$\frac{D_f(20)}{\pi^{23}\langle F, F\rangle} = -\frac{1}{15} \cdot \frac{2^{34}}{19!} \cdot \frac{\langle f, C_{18,3}\rangle}{\langle f, f\rangle}.$$

It remains to compute the ratio of the Petersson products. We know that $C_{18,3}$ is a cusp form in M_{18}, hence a multiple of F, so it suffices to compute its first Fourier coefficient c_1, which is equal to the ratio we are looking for.

We have $c_1 = \sum_{t=-2}^{2} P_{18,3}(t, 1)H(3, 4-t^2)$. Cohen has compiled in his paper a table of the values of H; from there, $H(3, N) = -1/252$, $-2/9$ and $-1/2$ for $N = 0$, 3, 4 respectively. The values of P can be found using any tool that can compute expansions (or in principle by hand – in our case, we used Wolfram Alpha), and they are -36, 45 and 11628 for $t = 0$, ± 1 and ± 2 respectively. We get $c_1 = -660/7$, and our ratio thus evaluates to

$$\frac{D_f(20)}{\pi^{23}\langle F, F\rangle} = \frac{11}{7} \cdot \frac{2^{36}}{19},$$

which agrees with the approximate value obtained by my supervisor.

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