Spectral Theory for $\mathrm{SL}_2\,\mathbb{Z}\backslash\mathbb{H}$

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1 Introduction

To motivate the need for the spectral theory we begin by investigating how the group $SL_2(\mathbb{Z}) := \{g \in M_2(\mathbb{Z}) : \det g = 1\}$ acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. The group (which we will denote Γ for brevity) is a discrete subgroup of $G := \mathrm{SL}_2\,\mathbb{R}$. G is important; one proves that it is precisely the group of isometries of $\mathbb{H}$.

Therefore Γ acts *properly discontinuously* on $\mathbb{H}$, i.e. for any two distinct points x, y in $\mathbb{H}$, there exist open neighbourhoods U, V containing x, y respectively such that the number of group elements g in Γ with $gU \cap V \neq \emptyset$ is finite. For such an action there is a notion of *fundamental domain*: a subset F of $\mathbb{H}$ such that

- $\mathbb{H} = \bigcup_{\gamma\in\Gamma} \gamma F$;
- There is an open set U so that $F = \bar{U}$;
- U and γU are either identical or disjoint.

We recall that a fundamental domain for the action of Γ on $\mathbb{H}$ is given by the set

$$F := \{z \in \mathbb{H} : -1/2 \leq \Re z \leq 1/2 \text{ and } |z| \geq 1\},$$

see ([2],Theorem 4.1.2, page 97). The images of F under Γ therefore tesselate $\mathbb{H}$; figure (1) shows a picture of F and its images – call these *tiles*.

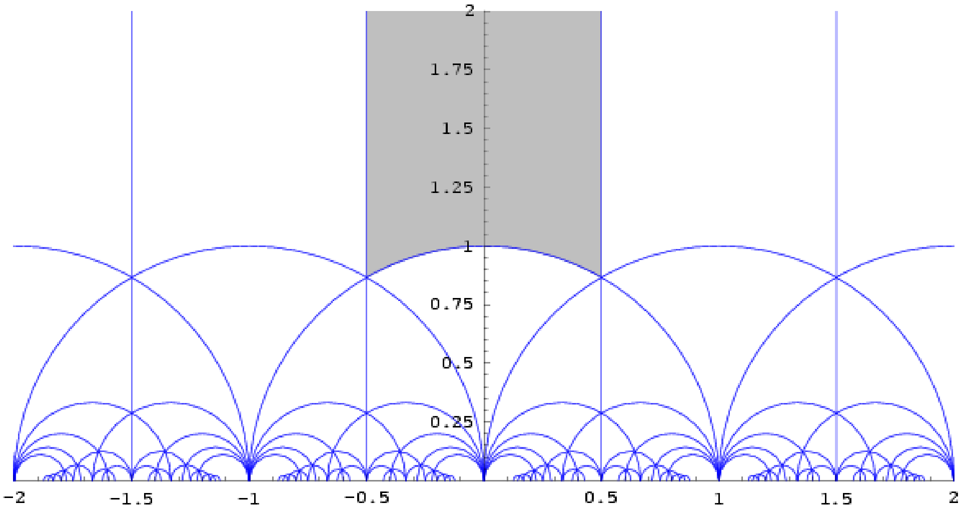

Figure 1: Shaded region is F.

A curve $v : [0, 1] \to \mathbb{H}$ will cross a finite number of tiles and therefore 'reflect' in F, inducing a curve $\hat{v} : [0, 1] \to F$. (If $v(t)$ is in γF then define $\hat{v}(t) = \gamma^{-1} v(t)$.) We are specifically interested in the family of line segments

$$v_\epsilon := \{x + i\epsilon : -1/2 \le x \le 1/2\}$$

for $\epsilon > 0$. How do these reflect in F?

Figure (2) shows how 200 evenly spaced points on v_ϵ reflect into F – for $\epsilon = 1/10, 1/25, 1/100$ and $1/250$. We observe that as ϵ gets smaller, the reflection of v_ϵ covers F more uniformly. It increasingly starts to look like 200 randomly points chosen according to a probability distribution that is 'bottom-heavy'. We recall the hyperbolic measure μ on $\mathbb{H}$ given in terms of the Lebesgue measure on $\mathbb{C}$ by

$$\mu(S) := \int_S \frac{dx\, dy}{y^2}$$

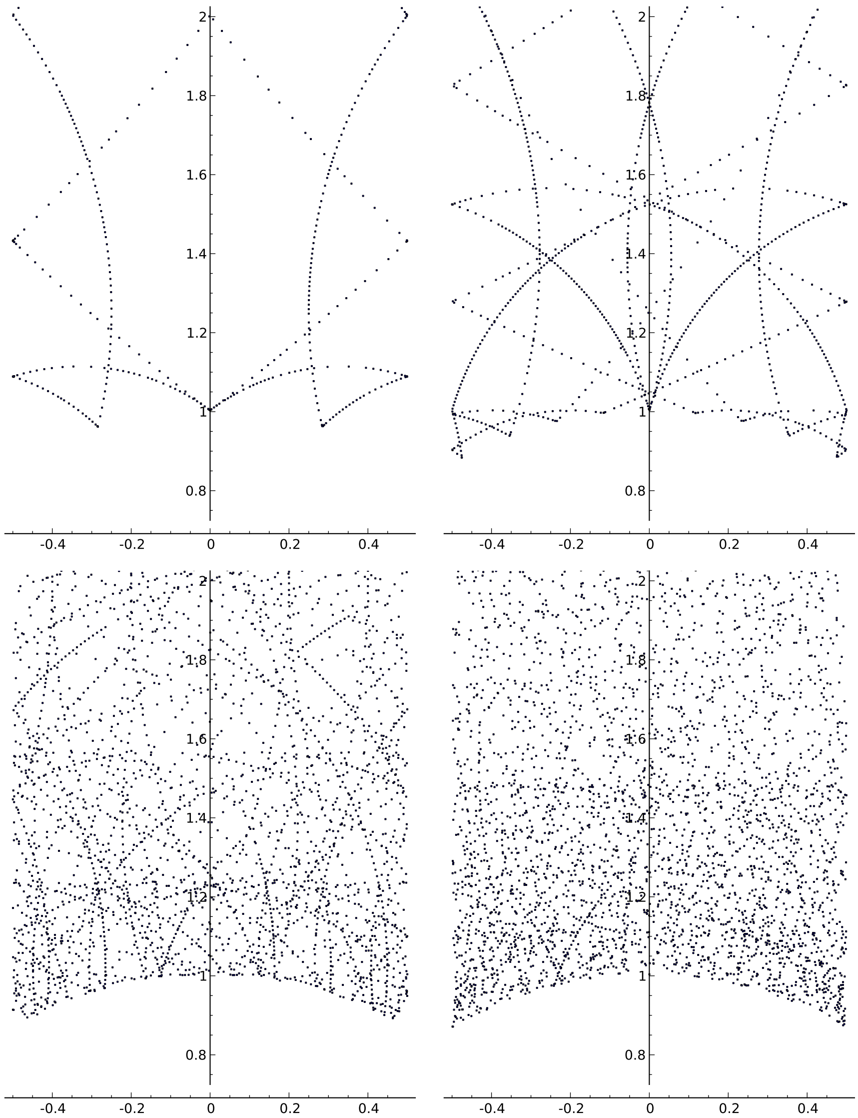

Figure 2: Clockwise from top-left $\epsilon = 1/10, 1/25, 1/250$ and $1/100$.

for any subset S of $\mathbb{H}$ that is measurable in the Lebesgue measure. We have $V := \mu(F) < \infty$. Furthermore the uniform measure on v_ϵ induces a measure μ_ϵ on F (it is a bit difficult to write down explicitly because $v_\epsilon(t)$ reflects depending on the tile it's in); again we have $\mu_\epsilon(F) = 1 < \infty$. Figure (2) now suggests that as $\epsilon \to 0$, μ_ϵ 'converges' to μ on F. We want to make this intuition precise. For this we need an appropriate notion of *convergence of* *measure.*

We may normalise μ and μ_ϵ and define

$$P(S) := \frac{\mu(S)}{\mu(F)} \qquad P_\epsilon(S) := \frac{\mu_\epsilon(S)}{\mu_\epsilon(F)}$$

These measures have total volume 1 and so are *probability measures*, and for probability measures we have a notion of *weak convergence of measure*:

Definition. *For probability measures P and $\{P_n\}_{n\geq 0}$ on a space X, let E be the expectation with respect to P and E_n the expectation with respect to P_n. We say P_n* converges weakly *to P as $n \to \infty$ if*

$$E_n f \to Ef \tag{1.1}$$

for every square-integrable (L^2) function f on X.

This is actually a stronger hypothesis than the standard one, which assumes f only to be bounded and continuous.

We may restrict ourselves to square-integrable *automorphic functions* on F, i.e. functions such that $f(\gamma z) = f(z)$ for $\gamma \in \Gamma$ and $z, \gamma z \in F$ (of course, this only applies to points on the boundary of F), since these functions are dense in $L^2(F)$. (Recall $L^2(F)$ is a normed space.) Such functions extend to automorphic functions on the whole of $\mathbb{H}$, i.e. functions in

$$\mathcal{L} := L^2(\Gamma\backslash\mathbb{H}).$$

This is a Hilbert space with normalised inner product given by

$$\langle f, g\rangle := \int_F f(z)\bar{g}(z)\ d\mu z$$

where μ is the hyperbolic measure defined above. For automorphic functions it is easy to write down $E_\epsilon f$ (the expectation w.r.t P_ϵ), since we needn't bother about reflection; it is simply

$$E_\epsilon f = \int_0^1 f(x + i\epsilon)\ dx$$

since v_ϵ has total length 1. So we set out to prove that $E_\epsilon f \to Ef$ as $\epsilon \to 0$, i.e.

$$\lim_{\epsilon \to 0} \int_0^1 f(x + i\epsilon)\ dx = \frac{1}{V} \int_F f(z)\ d\mu z$$

for all functions $f \in \mathcal{L}$. Equivalently,

$$\lim_{\epsilon \to 0} \int_0^1 f(x + i\epsilon)\overline{g(x + i\epsilon)}\ dx = \langle f, g \rangle \tag{1.2}$$

for all functions $f, g \in \mathcal{L}$. (1.2) is the formula we set out to prove.

For this we will require a spectral theory decomposing a function in $\mathcal{L}$ into eigenfunctions of the Laplace operator Δ

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),$$

which acts on the whole of $\mathcal{L}$, as we shall see (though this is not yet clear: for now Δ acts on twice-differentiable functions).

We shall see that $\mathcal{L}$ splits into the direct sum of two subspaces $\mathcal{C}$ and $\mathcal{E}$. On each subspace there is a different type of spectral theory: a 'discrete' version on $\mathcal{C}$ expressing functions as a countable sum of eigenfunctions (akin to the usual Fourier series expansion on the torus); and a 'continuous' part on $\mathcal{E}$ expressing functions as a continuous sum (integral) of eigenfunctions. We now define these spaces.

2 Definition of Spaces

At first we restrict ourselves to the subspace $\mathcal{B}$ of $\mathcal{L}$ of *smooth and bounded* automorphic functions, which is dense in $\mathcal{L}$. We shall split up $\mathcal{B}$ as a direct sum

$$\mathcal{B} = \mathcal{C} \oplus \mathcal{E}$$

and so

$$\mathcal{L} = \bar{\mathcal{B}} = \bar{\mathcal{C}} \oplus \bar{\mathcal{E}}$$

where $\bar{\cdot}$ denotes topological closure.

Definition of $\mathcal{C}$. Every smooth automorphic function $f(x+iy)$ is in particular invariant under the action of translations

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

and therefore has a Fourier series expansion in the variable x. Denote by $c_P f(y)$ the constant term in this expansion, so

$$c_P f(y) = \int_0^1 f(x+iy)\; dx$$

A function $f \in \mathcal{B}$ is called a *cusp form* if $c_P(f)(y)$ is identically zero. The set of cusp forms is a subspace of $\mathcal{B}$ denoted $\mathcal{C}$.

Definition of $\mathcal{E}$. Suppose φ is a smooth, compactly supported function on $\mathbb{R}^+$ (i.e. $\varphi \in C_c^\infty(\mathbb{R}^+)$). Then the function

$$f(z) := \varphi(\Im z)$$

defines a smooth function on $\mathbb{H}$ that is invariant under Γ_∞, though not necessarily under Γ. To get an automorphic function, sum over all cosets

$$\Psi_\varphi(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi(\Im \gamma z). \tag{2.1}$$

This type of function is called a *pseudo-Eisenstein* series. The sum is well-behaved:

Lemma 2.1. *The sum defining Ψ_φ converges absolutely and uniformly on compacts.*

Proof. Let $C \in \mathbb{H}$ be the compact set in question. The support of φ is a compact subset of $(0, \infty)$, so is contained in an interval $[p, q]$ with $p > 0$. I claim that the number of $\gamma \in \Gamma_\infty \backslash \Gamma$ with $\Im(\gamma C) >= p$ is finite.

Indeed for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and $z \in C$ we have

$$\Im(\gamma z) = \frac{\Im z}{|cz + d|^2}.$$

z is set to be in the compact set C, so $r \leq |\Im z| \leq s$ and $t \leq |\Re z| \leq v$ say $(r, s, t, v > 0)$. The equation

$$|cz + d| \leq \delta$$

(any $\delta > 0$) is true only for a finite number of c, d dependent only on C: $|cz + d| \geq |\Im(cz + d)| = c|\Im(z)| \geq cr$, giving a bound on the number of c, and $|cz + d| \geq |\Re(cz + d)| = |c\Re z + d|$, giving a bound on the number of d for every c. Pick $\delta = r/p$: then the number of (c, d) such that

$$\frac{\Im z}{|cz + d|} \geq \frac{r}{|cz + d|} \geq p$$

is a finite number dependent only on C. Because two elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$$

in Γ are in the same coset of Γ_∞, this proves the claim.

Given the claim we know that for $z \in C$ the number of terms in (2.1) is finite, the terms depending only on C; thus the sum is absolutely and uniformly convergent on C. □

We may improve the bound on the set $\sharp\{\gamma : \Im yz > Y\}$. We have

$$\sharp\{\gamma \in \Gamma_\infty \backslash \Gamma : \Im(\gamma z) > Y\} < 1 + \frac{k}{Y} \tag{2.2}$$

for a constant k. For a proof of this, see ([1], lemma 2.10, page 50).

The set of pseudo-Eisenstein series defines a subspace of $\mathcal{B}$ denoted $\mathcal{E}$. (By lemma 2.1, $\Psi_\varphi + \Psi_\phi = \Psi_{\varphi+\phi}$; and $\varphi + \phi$ is still in $C_c^\infty(\mathbb{R}^+)$.) $\mathcal{E}$ is not necessarily equal to $\mathcal{B}$; in fact we have that the orthogonal complement of $\mathcal{E}$ in $\mathcal{B}$ is precisely $\mathcal{C}$:

Lemma 2.2. *A function $f \in \mathcal{B}$ is a cusp form if and only if $\langle f, \Psi_\varphi \rangle = 0$ for all pseudo-Eisenstein series Ψ_φ.*

Proof. Compute the inner product

$$V \cdot \langle f, \Psi_\varphi \rangle = \int_F f(z) \cdot \sum_\gamma \varphi(\gamma z)\ d\mu z = \sum_\gamma \int_F f(z) \cdot \varphi(\gamma z)\ d\mu z \tag{2.3}$$

we may swap sum and integral by absolute convergence. Using the fact that elements of Γ are isometries ($\Gamma \subset G$) the integral becomes

$$\sum_\gamma \int_{\gamma F} f(z)\varphi(z)\ d\mu z = \int_{0 \le \Re(z) \le 1} f(z)\varphi(z)\ d\mu z$$

which (letting $z = x + iy$) we evaluate as

$$\int_0^\infty \int_0^1 f(x+iy)\varphi(y)\ dx\ \frac{dy}{y^2} = \int_0^\infty c_P f(y) \cdot \varphi(y)\ \frac{dy}{y^2}.$$

Since $c_P f(y)$ is smooth, it is either identically zero or nonvanishing on an interval $[a, b]$. In the former case, it is a cusp form; in the latter, we may choose a smooth, compactly supported φ that is greater or equal to 1 on $[a, b]$ such that the last integral does not vanish. □

3 Discrete Part of Spectrum

To find the spectral resolution of Δ on $\mathcal{C}$ we'll use functional analysis. The spectrum will arise from the application of one of the main results in the theory of bounded linear operators on Hilbert spaces, the *Hilbert-Schmidt theorem*. This gives the spectral resolution for a special class of bounded linear operators on a Hilbert space: the compact, self-adjoint bounded linear operators.

We recall that a *self-adjoint* operator T on a Hilbert space X is such that $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in X$; it is *compact* if the image of the unit ball $B(1) = \{f \in X : \|f\| \leq\}$ is *pre-compact*, i.e. the closure of $TB(1)$ is compact.

Theorem 3.1 (Hilbert-Schmidt)**.** *Suppose $L \neq 0$ is a self-adjoint compact bounded operator on a Hilbert space H. Then*

1. *if $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and $L - \lambda$ is not invertible, then λ is an eigenvalue of L (L 'has pure point spectrum', see below);*

2. *the eigenspaces of L have finite dimension;*

3. *the eigenvalues of L can accumulate only at zero;*

4. *one of* $\pm\|L\|$ *is an eigenvalue of* L*;*

5. *The range of* L *in* H *is spanned by eigenfunctions of* L*; if* $\{u_j\}_{j\geq 0}$ *is any maximal orthonormal system of eigenfunctions of* L *in* H*, then any* f *in the range of* L *has an absolutely and uniformly convergent series representation*

$$f(z) = \sum_{j\geq 0} \langle f, u_j \rangle u_j(z)$$

Proof. See ([1], theorem A.10, page 189) □

We would like to apply Hilbert-Schmidt to Δ acting on $\mathcal{L}$. However we haven't yet defined Δ as acting on the whole of $\mathcal{L}$: and even if we restrict ourselves to smooth functions f (for which Δf makes sense), it is not obvious that if f is square-integrable, then Δf is too. So this is not possible yet. But certainly Δ acts on the subspace $\mathcal{D}$ of $\mathcal{L}$, defined by

$$\mathcal{D} := \{f \in \mathcal{B} : \Delta f \in \mathcal{B}\}.$$

This subspace is dense in $\mathcal{L}$. We will prove that Δ is self-adjoint and non-positive (lemma 3.2) on $\mathcal{D}$, and therefore that it has a unique extension to the whole of $\mathcal{L}$, the *Friedrichs extension* (lemma 3.4).

Lemma 3.2. Δ *is self-adjoint on* $\mathcal{D}$*.* $-\Delta$ *is nonnegative on* $\mathcal{D}$*.*

Proof. By Stokes' theorem

$$\int_F \Delta f \bar{g}\ d\mu z = \int_F \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) fg\ dx\ dy = \int_F \nabla f \overline{\nabla g}\ dx\ dy + \int_{\partial F} \frac{\partial f}{\partial n} \bar{g}\ dl$$

where the boundary ∂F is piecewise smooth, $\Delta f = [\partial f/\partial x, \partial f, \partial y]$ is the gradient of f, dl is the euclidean length element and $\partial f/\partial n$ is the outward normal derivative. We may write the boundary integral in hyperbolic invariant form

$$\int_{\partial F} \frac{\partial f}{\partial n} \bar{g} \ dl = \int_{\partial F} \frac{\partial f}{\partial \mathbf{n}} \bar{g} \ d\mathbf{l}$$

where $\partial/\partial\mathbf{n} := y\partial/\partial n$ and $d\mathbf{l} := y^{-1}dl$. By invariance, equivalent parts of the boundary cancel each other out and so this integral vanishes. So

$$\langle -\Delta f, g\rangle = \int_F \nabla f \overline{\nabla g} \ dx \ dy = \langle f, \Delta g\rangle$$

so Δ is self-adjoint, and

$$\langle -\Delta f, f\rangle = \int_F |\nabla f|^2 \ dx \ dy \geq 0$$

so $-\Delta$ is nonnegative. □

Corollary 3.3. *Eigenvalues of Δ are real and nonnegative.*

Lemma 3.4 (Friedrichs Extension)**.** *Let H be a Hilbert space and G a dense subset of H. Suppose T is a linear operator defined on H that is nonnegative and self-adjoint. Then T extends to the whole of G.*

We now have a legitimate operator Δ acting on the whole of $\mathcal{L}$ that is what we think it is when acting on smooth functions. But unfortunately, we cannot apply Hilbert-Schmidt: Δ is not compact. Instead, our approach will be to construct a compact, self-adjoint bounded linear operator $\hat{L}$ with dense range; and with a complete orthonormal system of eigenfunctions that are also eigenfunctions of Δ.

Spectral Theory and Resolvents. $\hat{L}$ will be defined in terms of the resolvents of Δ. The resolvents for a linear operator T is the family of operators $(T - \lambda)^{-1}$ for $\lambda \in \mathbb{C}$. Of course, this operator isn't necessarily defined for all λ; the set of λ for which $T - \lambda$ is *not* invertible is called the *spectrum* $\sigma(T)$

of T. The spectrum obviously includes all eigenvalues, the set of which is called the *point spectrum* of T.

The spectrum is central to spectral theory. The Hilbert-Schmidt theorem consists essentially in proving that for a linear operator satisfying the hypotheses, the spectrum consists only of the point spectrum and possibly zero. The spectrum is generally easier to analyse than the point spectrum; for example, every bounded linear operator has compact spectrum. In particular, there exists λ so that $(T-\lambda)^{-1}$ is defined (take $|\lambda|$ sufficiently big).

We define $R_s := (\Delta - s(s-1))^{-1}$, when possible. (It will be clear shortly why the resolvent is indexed by s rather than $\lambda = s(s-1)$.) This will turn out to be an *invariant integral operator*, a special type of linear operator with several useful and important properties.

Properties of Invariant Integral Operators. An *integral operator* L is defined by

$$(Lf)(z) := \int_{\mathbb{H}} k(z,w) f(w) d\mu w \ ,$$

where $d\mu$ is the standard hyperbolic measure on $\mathbb{H}$ and $k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is a given function called the *kernel* of L. We don't (yet) assume that f is automorphic, but we need that f and k are such that the integral converges absolutely.

For an *invariant* integral operator we require in addition that

$$k(gz, gw) = k(z,w), \text{ for all } g \in G.$$

Since G is precisely the group of isometries on $\mathbb{H}$, this means that k is a *point-pair invariant*: it depends only on the hyperbolic distance $\rho(z,w)$between z and w. So we may write $k(z,w) = k(\rho(z,w))$; L is a convolution.

Quick Digression on Point-Pair Invariants. It will be useful to write point-pair invariants not in terms of the hyperbolic distance itself, but in terms of a different point-pair invariant $u(z, w)$, defined by

$$\cosh \rho(z, w) = 1 + 2u(z, w).$$

Since

$$\rho(z, w) = \log \frac{|z - \bar{w}| + |z - w|}{|z - \bar{w}| - |z - w|}$$

(where $|\cdot|$ is the Euclidean distance), we have

$$u(z, w) = \frac{|z - w|^2}{4\Im z \Im w}.$$

So we may write $k(z, w) = k(u(z, w))$. This will be used later.

Back to invariant integral operators. These operators are closely connected with Δ. We have

Lemma 3.5. *The invariant integral operators commute with Δ.*

Proof. We will use *geodesic polar coordinates.* These are derived from the Cartan decomposition of G

$$G = KAK$$

where K is the set of rotations and A the set of diagonal matrices,

$$K = \left\{ k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \leq \theta < \pi \right\}$$

$$A = \left\{ a(\alpha) = \begin{pmatrix} \sqrt{\alpha} & \\ & 1/\sqrt{\alpha} \end{pmatrix} : \alpha > 0 \right\}$$

(To see this multiply $g \in G$ on the left by $k_1 \in K$ to bring g to a symmetric matrix $g_1 = k_1 g$; then by conjugation in K the symmetric matrix g_1 can be brought to a diagonal matrix $a = kg_1k^{-1}$.)

G acts transitively on $\mathbb{H}$ so every $z \in \mathbb{H}$ may be written in the form $z = gi$ for some $g \in G$. Write $g = k(\phi)a(e^{-r})k(\theta)$ $(r \geq 0)$; then $z = k(\phi)e^{-r}i$ since K fixes i. Note $\rho(gi, i) = \rho(k(\phi)e^{-r}i, i) = \rho(e^{-r}i, i) = r$, so r is the hyperbolic distance from i to gi. The is the first version of the geodesic polar coordinates. The second version is given in terms of u instead of r; recall that $\cosh r = 1 + 2u$. In geodesic polar coordinates,

$$\Delta = u(u+1)\frac{\partial^2}{\partial u^2} + (2u+1)\frac{\partial}{\partial u} + \frac{1}{16u(u+1)}\frac{\partial^2}{\partial \phi^2}.$$

We our now set up to prove the lemma. Suppose L is an invariant integral operator given by invariant kernel k, so $k(z, w)$ is a smooth point-pair invariant on $\mathbb{H} \times \mathbb{H}$. Then we have

$$\Delta_z k(z, w) = \Delta_w k(z, w).$$

Indeed, using geodesic polar coordinates for w, we get

$$\Delta_z k(z, w) = u(u+1)k''(u) + (2u-1)k'(u),$$

and using geodesic polar coordinates for z we get the same expression for $\Delta_w k(z, w)$.

So

$$\begin{aligned}
\Delta L f(z) &= \int_{\mathbb{H}} \Delta_z k(z, w) f(w)\ d\mu w = \int_{\mathbb{H}} \Delta_w k(z, w) f(w)\ d\mu w \\
&= \langle \Delta_w k(z, \cdot), f\rangle = \langle k(z, \cdot), \Delta_w f\rangle \quad \text{since } \Delta_w \text{ is symmetric } (*) \\
&= \int_{\mathbb{H}} k(z, w)\Delta_w f(w)\ d\mu w = L\Delta f(z).
\end{aligned}$$

$*$: The proof that Δ_w is symmetric goes as in the proof of (3.2). $\square$

Theorem 3.6. *Every eigenfunction of Δ is also an eigenfunction of all invariant integral operators with kernel $k(u)$ in $C_0^\infty(\mathbb{R}^+)$. Conversely, if f is an eigenfunction of all such integral operators, then f is also an eigenfunction of Δ.*

Proof. See ([1], theorems 1.14 and 1.15, page 30). $\square$

If we restrict the domain of L to automorphic functions then we may write

$$(Lf)(z) = \int_F K(z,w)f(w)d\mu w,$$

where F is our standard fundamental domain for Γ on $\mathbb{H}$ and

$$K(z,w) := \sum_{\gamma\in\Gamma} k(z,\gamma w).$$

This new kernel is called the *automorphic kernel.*

These operators are especially convenient on $\mathcal{C}$, as we can form the 'compact version' of any such operator subject to certain conditions on the kernel, as we shall see later. This is an essential ingredient of the proof and the reason why we restrict ourselves to $\mathcal{C}$ in the spectral resolution of Δ, rather than the whole of $\mathcal{L}$.

We check that L does indeed act on $\mathcal{C}$:

Lemma 3.7. *An invariant integral operator L maps the subspace $\mathcal{C}$ of $\mathcal{L}$ to itself:*

$$L : \mathcal{C} \to \mathcal{C}.$$

Proof. Let $f \in \mathcal{C}$ and $g = Lf$. Then

$$\begin{aligned} c_P g(y) &= \int_0^1 g(z+t)\ dt = \int_0^1 \int_{\mathbb{H}} k(z+t,w) f(w)\ d\mu w\ dt \\ &= \int_0^1 \int_{\mathbb{H}} k(z+t,w+t) f(w+t)\ d\mu w\ dt \\ &= \int_{\mathbb{H}} k(z,w) \int_0^1 f(w+t)\ dt\ d\mu w \qquad \text{since } k \text{ is invariant} \\ &= \int_{\mathbb{H}} k(z,w)\ c_P f(\Im w)\ d\mu w = 0, \end{aligned}$$

since $c_P f = 0$. □

Before showing how to construct the compactification $\hat{L}$ of L, we prove that $-R_s$ is an invariant integral operator.

The Resolvent is an Invariant Integral Operator. Let

$$G_s(u) := \frac{1}{4\pi} \int_0^1 (\xi(1-\xi))^{s-1} (\xi+u)^{-s} d\xi. \tag{3.1}$$

Suppose $s \in \mathbb{C}$ with $\Re s > 1$. Let $-T_s$ be the invariant integral operator with kernel G_s, i.e.

$$-(T_s f)(z) := \int_{\mathbb{H}} G_s(u(z,w)) f(w) d\mu w.$$

(with u as defined above). Then

Theorem 3.8. *If f is smooth and bounded on $\mathbb{H}$, then*

$$(\Delta + s(1-s)) T_s f = f.$$

Proof. See ([1], theorem 1.17, page 32). □

So T_s is the right inverse of $\Delta + s(1-s)$ and so coincides with R_s when R_s is defined (and $\Re s > 1$).

The kernel G_s has the following properties, which we will need later:

Lemma 3.9. *The integral (3.1) defining G_s converges absolutely for $\Re(s) = \sigma > 0$. It gives a function $G_s(u)$ on $\mathbb{R}^+$ which satisfies*

$$(\Delta + s(s-1))G_s = 0.$$

Moreover, $G_s(u)$ satisfies the following bounds:

$$G_s(u) = \frac{1}{4\pi}\log\frac{1}{u} + O(1) \qquad \text{as } u \to 0; \qquad (3.2)$$

$$G_s'(u) = -(4\pi u)^{-1} + O(1) \qquad \text{as } u \to 0; \qquad (3.3)$$

$$G_s(u) \ll u^{-\sigma} \qquad \text{as } u \to +\infty. \qquad (3.4)$$

Proof. See ([1] lemma 1.7, page 25). □

We need one final ingredient to define our operator: given an invariant integral operator L acting on $\mathcal{C}$, whose kernel k is smooth and compactly supported, we can form a *compact* invariant integral $\hat{L}$ on $\mathcal{C}$ with range identical to that of L.

Compactification of Invariant Integral Operators on $\mathcal{C}$. We will construct an integral operator that is defined by

$$(Tf)(z) = \int_F \kappa(z,w)f(w),$$

where the kernel κ is bounded on $F \times F$. Thus in particular

$$\int_F \int_F |\kappa(z,w)|^2 \; d\mu z \; d\mu w < \infty$$

i.e. $k \in L^2(F \times F)$. Thus the integral operator on F with kernel k is a *Hilbert-Schmidt integral operator,* and is therefore compact.

Lemma 3.10. *Hilbert-Schmidt integral operators are compact.*

Proof. See ([3], exercise 4.15, page 106) ☐

We would like to take $\kappa = K$, the automorphic kernel. But unfortunately K isn't bounded on $F \times F$, no matter how small we take the support of k. This is because as z, w approach infinity, the number of terms which count in $K(z,w) = \sum_\gamma k(z, \gamma w)$ grows to infinity. To get a bounded kernel κ we subtract from $K(z,w)$ the *principal part*

$$H(z,w) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{-\infty}^{+\infty} k(z, \gamma w + t) dt$$

and we define

$$\hat{K}(z,w) := K(z,w) - H(z,w).$$

This new automorphic kernel defines an operator $\hat{L}$ on $\mathcal{L}$ provided we can prove that this integral always converges. This is the case:

Lemma 3.11. *The function $w \mapsto H(z,w)$ is bounded and invariant under Γ, i.e.*

$$H(z, \cdot) \in \mathcal{B}.$$

In particular the integral defining $(Lf)(z)$ converges for all z.

Proof. We estimate

$$H(z,w) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{-\infty}^{\infty} k(z, t + \gamma w)\ dt.$$

$k(u)$ has compact support, so the range of integration is restricted by

$$u(z, t + \gamma w) = \frac{|z - t - \gamma w|^2}{4 \Im z \Im \gamma w} \leq c$$

for a constant c, so $|z - t - \gamma w|^2 \leq 4c \Im z \Im \gamma w$. This implies that $|\Im(z) - \Im(\gamma w)|$

is bounded. So the integral is bounded by $O(\Im(z))$, and by formula (2.2) the number of terms that count is bounded by $1 + O(1/\Im(z)$. So

$$H(z, w) \leq \left(1 + O\left(\frac{1}{\Im z}\right)\right) \cdot O(\Im z) = O(\Im z).$$

In particular $H(z, \cdot)$ is bounded. It is clearly automorphic. $\square$

We claim that $\hat{L}$ is the 'compactified' operator we want: it is compact and it acts like L on $\mathcal{C}$:

Lemma 3.12. *For $f \in \mathcal{C}$ we have $Lf = \hat{L}f$.*

Lemma 3.13. *The kernel $\hat{K}(z, w)$ is bounded on $F \times F$; therefore $\hat{L}$ is a compact operator.*

Proof of 3.12. We shall prove that $H(z, \cdot)$ is orthogonal to the space $\mathcal{C}$, i.e.

$$\langle H(z, \cdot), f \rangle = 0 \quad \text{if } f \in \mathcal{C}.$$

We do this directly by unfolding the integral:

$$\begin{aligned}
\langle H(z, \cdot), f \rangle &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_F \int_{-\infty}^{\infty} k(z, \gamma w + t)\, dt \bar{f}(w)\, d\mu z \\
&= \int_0^\infty \int_0^1 \int_{-\infty}^{\infty} k(z, w + t) \bar{f}(w)\, \frac{dx\, dy}{y^2} \quad \text{as in lemma (2.2)} \\
&= \int_0^\infty \left(\int_{-\infty}^{\infty} k(z, t + iy)\, dt \right) \left(\int_0^1 \bar{f}(x + iy)\, dx \right) \frac{dy}{y^2} \\
&= \int_0^\infty \left(\int_{-\infty}^{\infty} k(z, t + iy)\, dt \right) c_P(\bar{f})(y)\, \frac{dy}{y^2} = 0
\end{aligned}$$

(letting $w = x + iy$), since $c_P(\bar{f}) = \overline{c_P(f)} = 0$. $\square$

Proof of 3.13. This uses a similar estimate to the one in the previous proof. See ([1], proposition 4.5, page 67). $\square$

Note that 3.12 crucially requires L to be acting on $\mathcal{C}$, rather than the whole of $\mathcal{L}$ - which is why our spectral resolution restricts to this smaller space.

For simplicity we have assumed that the kernel k is compactly supported; however the key results above (3.12, 3.13, 3.11) remain true if we assume k decays sufficiently rapidly -

$$k(u), k'(u) \ll (u+1)^{-2}. \tag{3.5}$$

Granted this we now have all of the ingredients to prove the spectral resolution of Δ on $\mathcal{C}$.

Spectral Resolution of Δ on $\mathcal{C}$. As promised we define L in terms of the resolvent. Our first guess will be $L = R_s$ for $s >= 2$. The range is dense in $\mathcal{C}$: given $f \in \mathcal{C} \cap \mathcal{D}$ (dense in $\mathcal{C}$)let $g := (\Delta + s(s-1))f$ – then $f = R_s g$. Since $s >= 2$, the kernel decays sufficiently rapidly as $u \to \infty$ (condition 3.5, lemma 3.9). But unfortunately we cannot form the compactification of L because G_s is singular at zero (lemma 3.9).

To solve this problem we take $L = R_s - R_a$, $a > s \geq 2$ instead. This kills the singularity – the kernel $k(u) = G_a(u) - G_s(u)$ is smooth; and it still decays according to 3.5. So we may form the compactified operator $\hat{L}$. The range is still dense: using the *Hilbert Formula*

$$L = R_s - R_a = (s(1-s) - a(1-a)R_s R_a$$

(proof: apply R_s to the obvious identity $I - (\Delta - s(1-s))R_a = (\Delta - a(1 - a) - (\Delta - s(s-1)))R_a = (s(1-s) - a(1-a))R_a$), we let

$$g = (s(1-s) - a(1-a))^{-1}(\Delta + a(1-a))(\Delta + s(1-s))f$$

for $f \in \mathcal{C} \cap \mathcal{D}$.

Summarising, the compactified operator $\hat{L}$ has the following properties:

(a) It is compact;

(b) It is self-adjoint, because the kernel is real;

(c) Its range is dense;

So we may apply the Hilbert-Schmidt theorem to $\hat{L}$ and deduce

Proposition 3.14. *$\mathcal{C}$ is spanned by eigenfunctions of $\hat{L}$. The eigenvalues are bounded and accumulate only at zero, and the eigenspaces are finite-dimensional.*

Let $\{u_j\}_{j\geq 0}$ be any complete orthonormal set of eigenfunctions of $\hat{L}$. Then any $f \in \mathcal{C}$ has expansion

$$f = \sum_{j\geq 0} \langle f, u_j \rangle u_j(z)$$

which converges absolutely and uniformly on compacts.

We claim that there is a complete orthonormal system $\{u_j\}_{j\geq 0}$ of eigenfunctions of $\hat{L}$ that are also eigenfunctions of Δ. Indeed, let $\mathcal{C}_\lambda$ be the eigenspace of $\hat{L}$ corresponding to λ. Δ and $\hat{L}$ commute, since $\hat{L}$ is an integral operator (lemma 3.5); so Δ maps $\mathcal{C}_\lambda$ to itself. We know $\mathcal{C}_\lambda$ is finite dimensional and so the action of Δ on it is expressible as a matrix; it will be an Hermitian matrix because Δ is self-adjoint. By linear algebra, hermitian matrices are diagonalisable: in other words $\mathcal{C}_\lambda$ is spanned by eigenfunctions of Δ, say $\{u_{\lambda j}\}_j$. The set $\{u_{\lambda j}\}_{\lambda,j}$ ranging over all λ consists of simultaneous eigenfunctions of Δ and $\hat{L}$ and span every $\mathcal{C}_\lambda$ – so they span $\mathcal{C}$.

We conclude

Theorem 3.15 (Spectral Resolution of Δ on $\mathcal{C}$)**.** *$\mathcal{C}$ is spanned by eigenfunctions of Δ. Let $\{u_j\}_{j\geq 0}$ be any complete orthonormal set of eigenfunctions of Δ. Then any $f \in \mathcal{C}$ has expansion*

$$f = \sum_{j\geq 0} \langle f, u_j \rangle u_j(z)$$

which converges absolutely and uniformly on compacts.

4 Continuous Part

We spectrally decompose the pseudo-Eisenstein series using the *Mellin transform*, a variant of the Fourier transform, which we recall.

The Mellin Transform. For functions f in $L^1(\mathbb{R})$ we define the *Fourier transform* of f

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx.$$

If $\hat{f}$ is also in $L^1(\mathbb{R})$ then we have the *Fourier inversion theorem*:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

see ([4], 9.11, page 185). We may rewrite this as the identity

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-it\xi} dt \right) e^{i\xi x} d\xi \tag{4.1}$$

by replacing ξ with $\xi/(2\pi)$.

If we assume that f is compactly supported ($f \in C_c^\infty(\mathbb{R})$ then by the *Paley-Wiener theorem* (see [4], 19.3, page 375), $\hat{f}(\xi)$ extends to an *entire* function which is of rapid decay along horizontal lines. So by Cauchy's theorem we have

$$f(x) = \int_{-\infty+i\tau}^{\infty+i\tau} \hat{f}(\xi) e^{2\pi i \xi x} d\xi,$$

for any fixed τ, since the integral along vertical line segments $[x, x+i\tau]$ tends to zero as x tends to $\pm\infty$.

A variant of this is the *Mellin transform.* Suppose $F \in C_c^\infty(0,\infty)$ and let $f(x) = F(e^x)$. (Then $f \in C_c^\infty(0,\infty)$ too, so in $L^1(\mathbb{R})$.) Write $y = e^x$ and $r = e^t$. Then the identity becomes

$$F(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_0^\infty F(r) r^{-i\xi} \frac{dr}{r} \right) y^{i\xi} d\xi.$$

So we define the *Mellin transform* $\mathcal{M}F$ of F as

$$\mathcal{M}F(i\xi) := \int_0^\infty F(r) r^{-i\xi} \frac{dr}{r}$$

and as above we extend to the whole complex plane ($s \in \mathbb{C}$)

$$\mathcal{M}F(s) := \int_0^\infty F(r) r^{-s} \frac{dr}{r}.$$

Identity (4.1) becomes

$$F(y) = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} \mathcal{M}F(s) y^s ds$$

and like before this remains true integrating along any vertical line $[\sigma - i\infty, \sigma + i\infty]$, i.e.

$$F(y) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}F(s) y^s ds.$$

We will apply the Mellin inversion to the $\varphi \in C_c^\infty(\mathbb{R}^+)$ defining the pseudo-Eisenstein series

$$\Psi_\varphi(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\Im(\gamma z)).$$

By Mellin inversion

$$\varphi(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) y^s ds$$

so

$$\Psi_\varphi(z) = \frac{1}{2\pi i}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)(\Im(\gamma z))^s ds.$$

We wish to swap the order of the double integral (sum and integral). We can do this if it is absolutely convergent, by Fubini's theorem; this is the case for σ sufficiently large, say $\sigma > 1$. We obtain

$$\Psi_\varphi(z) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s) E_s(z) ds$$

where E_s is the *Eisenstein series* defined for $\Re(s) > 1$ by

$$E_s(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s.$$

The series is defined for $\Re(s) > 1$ because it is then absolutely convergent; but this is not necessarily the case for $\Re(s) \leq 1$. But in fact E_s has meromorphic continuation to the whole complex plane:

Theorem 4.1. *$E_s(z)$ has meromorphic continuation to the whole complex plane as a function of s. It has a single pole in the the half plane $\Re(s) \geq 1/2$ at $s = 1$, and this is a simple pole. Furthermore the residue at* 1

$$\operatorname{Res}_{s=1} E_s(z)$$

is a constant independent of z.

Proof. See ([2], corollary 7.2.11 and proof, pages 286-287). □

These Eisenstein series are the basic eigenpackets with which we decompose the pseudo-Eisenstein series: since

$$\Delta(y^s) = s(s-1)y^s$$

we have

$$\Delta E_s = s(s-1)E_s.$$

This is the first version of the spectral resolution – expressing Ψ_φ in terms of a continuous sum (integral) of Eisenstein series (the eigenpackets) with coefficients given in terms of $\mathcal{M}\varphi$. But we would like to express the coefficients in terms of Ψ_φ itself, not just the constituent function φ. This is what we do next.

First we use theorem (4.1) to move the line of integration to the left to $\sigma = 1/2$, for reasons that will be clear shortly. By the Residue theorem and (4.1)

$$\Psi_\varphi(z) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s(z)ds + \mathrm{Res}_{s=1}(E_s \cdot \mathcal{M}\varphi(s)).$$

Recall $c_P(f)$. We shall find the spectral resolution in terms of $\mathcal{M}c_P(\Psi_\varphi)$. There will be two steps to this:

Step 1. *For $f \in C_c^\infty(\Gamma\backslash\mathbb{H})$*

$$\int_F E_s(z)f(z)\ d\mu z = \mathcal{M}c_P(f)(1-s)$$

Step 2. *For a pseudo-Eisenstein series* Ψ_φ

$$\int_F E_s(g)\Psi_\varphi(z)\ d\mu z = \mathcal{M}\varphi(1-s) + c_s \mathcal{M}\varphi(s),$$

where c_s *is meromorphic in* s*.*

These two steps allow us to conclude that

$$\mathcal{M}c_P\Psi_\varphi(s) = \mathcal{M}\varphi(1-s) + c_{1-s}\mathcal{M}\varphi(s). \tag{4.2}$$

Formula (4.2) will allow us to deduce the final version of the spectral resolution from the first one – but first we prove the two steps.

Proof of Step 1.

$$\begin{aligned}\int_F E_s(z)f(z)dz &= \int_F \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s f(z)\ d\mu z\\ &= \int_{0\leq Re(z)\leq 1} \Im(z)^s f(z)\ d\mu z\\ &= \int_0^\infty \int_0^1 y^s f(x+iy)\ dx\ \frac{dy}{y^2}\\ &= \int_0^\infty y^s c_P f(y)\ \frac{dy}{y^2}\\ &= \int_0^\infty y^{-(1-s)} c_P f(y)\ \frac{dy}{y}\\ &= \mathcal{M}c_P f(1-s).\end{aligned}$$

□

Properties of Eisenstein Series. To prove Step 2 we derive some properties of the Eisenstein series. Δ commutes with the map $f \to c_P f$, since we may interchange order of differentiation and summation,

$$\Delta \int_0^1 f(x+iy)\,dx = \int_0^1 \Delta f(x+iy)\,dy,$$

because $f \in C_c^\infty(\mathbb{R}^+)$. Thus $c_P E_s$ is a function $u(y)$ of y satisfying

$$y^2 \frac{\partial^2}{\partial y^2} u(y) = s(s-1)u(y).$$

For $s \neq 1/2$ this has two linearly independent solutions y^s and y^{1-s}, and so for meromorphic functions a_s and c_s

$$c_P E_s = a_s y^s + c_s y^{1-s}.$$

By directly expanding the integral

$$c_P E_s(y) = \int_0^1 E_s(x+iy)\,dx$$

we deduce that $a_s = 1$. An important fact about E_s is that it has a 'universal property':

Theorem 4.2. *The equations*

$$\Delta w = s(s-1)w \qquad \left(y\frac{\partial}{\partial y} - (1-s)\right) c_P w = (2s-1)y^s$$

uniquely *determine* $w = E_s$.

Granted this, one readily checks that both E_s and $c_{1-s}^{-1} E_s$ satisfy the equations and so are identical by uniqueness. We obtain the *functional equation*

$$E_{1-s} = c_{1-s} E_s \tag{4.3}$$

Proof of Step 2. Proceeding as in the proof of Step 1,

$$\begin{aligned}
\int_F E_s(z)\Psi_\varphi(z)\ d\mu z &= \int_F E_s(z) \sum_{\Gamma_\infty\backslash\Gamma} \varphi(\Im\gamma z) d\mu z \\
&= \int_{0\leq\Re(z)\leq 1} E_s(z)\varphi(z)\ d\mu z \\
&= \int_0^\infty \int_0^1 E_s(x+iy)\varphi(y)\ dx\ \frac{dy}{y^2} \\
&= \int_0^\infty c_P(E_s)(y)\varphi(y)\ \frac{dy}{y^2}.
\end{aligned}$$

Substituting $c_P(E_s)(y) = y^s + c_s y^{1-s}$ we get

$$\int_0^\infty (y^{s-1} + c_s y^{-s})\ \frac{dy}{s} = \mathcal{M}\varphi(1-s) + c_s\mathcal{M}\varphi(s).$$

□

Spectral Resolution of Δ *on* $\mathcal{E}$. Finally we use formula (4.2) to obtain the spectral resolution in terms of $\mathcal{M}c_P\Psi_\varphi(s)$.

$$\begin{aligned}
&\Psi_\varphi(z) - \mathrm{Res}_{s=1}(\mathcal{M}\varphi(s)\cdot E_s(z)) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s(z)\ ds \\
&= \frac{1}{4\pi i}\left(\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s(z)\ ds + \int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(1-s)E_{1-s}(z)\ ds\right) \quad (*) \\
&= \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s(z) + \mathcal{M}\varphi(1-s)E_{1-s}(z)\ ds \\
&= \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s(z) + \mathcal{M}\varphi(1-s)c_{1-s}E_s(z)\ ds \quad \text{by (4.3)} \\
&= \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty} (\mathcal{M}\varphi(s) + c_{1-s}\mathcal{M}\varphi(1-s))E_s(z)\ ds \\
&= \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}c_P\Psi_\varphi(s)E_s(z)\ ds \\
&= \frac{1}{2\pi i}\int_{1/2+i0}^{1/2+i\infty} \mathcal{M}c_P\Psi_\varphi(s)E_s(z)\ ds
\end{aligned}$$

$$= \frac{1}{2\pi i} \int_{1/2+i0}^{1/2+i\infty} \langle \Psi_\varphi, E_{1-s} \rangle E_s(z)\ ds.$$

Note that the two integrals in the sum $(*)$ are the same for $\Re(s) = 1/2$ – this is why we moved the line of integration to the left before (i.e. to $\sigma = 1/2$).

Evaluation of the Residue Term. We finish this section by identifying the residue term $\mathrm{Res}_{s=1}(\mathcal{M}\varphi(s) \cdot E_s(z)) = \mathcal{M}\varphi(1) \cdot \mathrm{Res}_{s=1} E_s(z)$. We will use the fact that $\mathrm{Res}_{s=1} E_s(z)$ is *constant* in z, say k: see (4.1). To compute $\mathcal{M}\varphi 1$, we proceed as in the proof of step 1 and 2, but in reverse:

$$\begin{aligned}
\mathcal{M}\varphi(1) &= \int_0^\infty \varphi(y) y^{-1}\ \frac{dy}{y} \\
&= \int_0^\infty \int_0^1 \varphi(\Im(x+iy))\ dx\ \frac{dy}{y^2} \\
&= \int_{0 \le \Re z \le 1} \varphi(\Im z)\ d\mu z \\
&= \int_F \Psi_\varphi(z)\ d\mu z = \langle \Psi_\varphi, 1 \rangle.
\end{aligned}$$

Now constant functions are orthogonal to functions given by

$$\frac{1}{2\pi i} \int_{1/2+i0}^{1/2+i\infty} f(\Im(s)) E_s(g)\ ds$$

for f in $C_c^\infty(\mathbb{R}^+)$, so we have

$$\langle \Psi_\varphi, 1 \rangle = \langle \Psi_\varphi, 1 \rangle \langle k, 1 \rangle.$$

and so $k = 1$ if we assume that the hyperbolic measure is normalised. In this case the spectral theorem becomes

$$\Psi_\varphi(z) = \frac{1}{2\pi i} \int_{1/2+i0}^{1/2+i\infty} \mathcal{M} c_P \Psi_\varphi(s) E_s(z)\ ds + \langle \Psi_\varphi, 1 \rangle \cdot 1.$$

5 Spectral Theorem. Proof of Proposition

Combining the results of the previous two sections we obtain the Spectral Theorem:

Theorem 5.1 (Spectral Theorem)**.** *A function $f \in \mathcal{L}$ has spectral resolution in terms of eigenfunctions of Δ*

$$f = \sum_{j \geq 0} \langle f, u_j \rangle u_j + \int_{1/2+i0}^{1/2+i\infty} \langle f, E_{1-s} \rangle E_s \; ds + \langle f, 1 \rangle \cdot 1,$$

where the E_s are the Eisenstein series and the u_j a complete orthonormal system of eigenfunctions of Δ in $\mathcal{C}$. The sum and integral converge absolutely and uniformly on compacts.

We recall the proposition we set out to prove in the beginning.

Proposition 5.2. *Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathcal{L}$ induced by the normalised hyperbolic measure on $\mathbb{H}$:*

$$\langle f, g \rangle := \frac{1}{V} \int_F f(z) g(z) \; d\mu z$$

where V is the area of F. Then

$$\langle f, g \rangle = \lim_{y \to 0^+} \int_0^1 f(x+iy) \overline{g(x+iy)} \; dx.$$

Proof. We spectrally decompose $h := f\bar{g}$:

$$h = \sum_{j \geq 0} \langle h, u_j \rangle u_j + \int_{1/2+i0}^{1/2+i\infty} \langle h, E_{1-s} \rangle E_s \; ds + \langle h, 1 \rangle \cdot 1$$

and integrate each of the three terms on the right hand side separately. The cusp forms integrate out to zero, by definition:

$$\int_0^1 \sum_j \langle f, u_j \rangle u_j(x+iy)\ dx = \sum_j \langle f, u_j \rangle \int_0^1 u_j(x+iy)\ dx = 0,$$

since every term is zero.

For the pseudo-Eisenstein series we have

$$\begin{aligned}\int_0^1 \int_{1/2+i0}^{1/2+i\infty} \langle f, E_{1-s} \rangle E_s(x+iy)\ ds\ dx &= \int_{1/2+i0}^{1/2+i\infty} \langle f, E_{1-s} \rangle \int_0^1 E_s(x+iy)\ dx\ ds \\ &= \int_{1/2+i0}^{1/2+i\infty} \langle f, E_{1-s} \rangle c_P E_s(y)\ ds\end{aligned}$$

We know $c_P E_s(y) = y^s + c_s y^{1-s}$ so the integral becomes

$$\int_{1/2+i0}^{1/2+i\infty} \langle f, E_{1-s} \rangle (y^s + c_s y^{1-s})\ ds$$

We will show that this integral is bounded by $\sqrt{y}$ and therefore tends to zero as y does. Indeed if $\Re(s) = 1/2$, then $|y^s| = |y^{1-s}| = \sqrt{y}$. Furthermore by the functional equation(4.3),

$$E_{1-s} = c_{1-s} E_s = c_{1-s} c_s E_{1-s}$$

and so

$$c_s c_{1-s} = 1.$$

Since $E_{\bar{s}} = \overline{E_s}$ we have $\overline{c_s} = c_{\bar{s}}$ and so $|c_{1/2+it}|^2 = 1$ for real t, in other words $|c_s| = 1$ if $\Re(s) = 1/2$. Therefore

$$|y^s + c_s y^{1-s}| \leq |y^s| + |c_s||y^{1-s}| \leq 2\sqrt{y}.$$

Thus the integral is bounded by

$$2\sqrt{y}\int_{1/2+i0}^{1/2+i\infty} |\langle f, E_{1-s}\rangle| \; ds$$

which tends to zero as y does, as claimed.

Finally the constant term integrates to

$$\lim_{y\to 0}\int_0^1 \langle h, 1\rangle(x+iy)\; dx = \langle h, 1\rangle = \langle f\bar{g}, 1\rangle = \langle f, g\rangle.$$

Putting these three parts together we get the result. □

References

[1] Henryk Iwaniec, *Spectral Methods of Automorphic Forms.* American Mathematical Society, 2002.

[2] Toshitsune Miyake, *Modular Forms.* Springer, 1989.

[3] Walter Rudin, *Functional Analysis.* McGraw-Hill, 1973.

[4] Walter Rudin, *Real and Complex Analysis.* third edition, McGraw-Hill, 1987.