THE CATEGORY OF COMPLEXES

Let $\mathcal{C}$ be an abelian category.

Def. the category $Ch^{\cdot}(\mathcal{C})$ of cochain complexes over $\mathcal{C}$ is defined as follows.

- objects are pairs $(X^i, d^i)_{i\in\mathbb{Z}}$ such that $d^i : X^i \longrightarrow X^{i+1}$ and $d^{i+1}\circ d^i = 0 \quad \forall i\in\mathbb{Z}$

  (the sequence $\cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \longrightarrow \cdots$ is a complex)
- morphisms are collections $(f^i)_{i\in\mathbb{Z}}$ where $f^i \in \mathrm{Hom}_{\mathcal{C}}(X^i, Y^i)$ such that

$$d_{Y^i}\circ f^i = f^{i+1}\circ d_{X^i} \qquad \forall i$$

Frequently we denote an object by $X^{\cdot}$

There exists also a category $Ch_{\cdot}(\mathcal{C})$ of chain complexes over $\mathcal{C}$ where differentials lower the degree by 1 rather than raising. The category $Ch_{\cdot}(\mathcal{C})$ is the opposite category of $Ch^{\cdot}(\mathcal{C})$

Attention: we will use cochain complexes but Weibel's uses chain complexes.

Proposition: $Ch^{\cdot}(\mathcal{C})$ is abelian and if $f^{\cdot} : X^{\cdot} \longrightarrow Y^{\cdot}$

$$\ker(f^{\cdot}) = (\ker f^i)_{i\in\mathbb{Z}}$$
$$\mathrm{coker}(f^{\cdot}) = (\mathrm{coker}\, f^i)_{i\in\mathbb{Z}}$$

Proof: straightforward check.

Warning: it is not true that a complex with injective/projective terms is injective/projective in $Ch^{\cdot}(\mathcal{C})$.

Def: the n-th cohomology functor $Ch^{\cdot}(\mathcal{C}) \longrightarrow \mathcal{C}$ is the functor that sends $X^{\cdot}$ to the n-th cohomology:

$$H^n(X^{\cdot}) = \frac{\ker(d^n_X)}{\mathrm{Im}(d^{n-1}_X)} \qquad \text{(notation: } X/Y = \mathrm{coker}(Y \longrightarrow X))$$

with the natural morphisms. The collection $(H^i(X), d^i)_{i\in\mathbb{Z}}$ is a complex in $Ch^{\cdot}(Ab)$

Delicate point: this is not actually well-defined because kernels are not unique, they are defined only up to unique isomorphisms. We need a rule attaching a specific choice of a kernel and a cokernel to every morphism in $\mathcal{C}$. This turns out to be ok for R-modules but in general is dangerous because $\mathrm{Hom}(\ker f, x)$ wouldn't be a set.

Def: a morphism in $Ch^{\cdot}(\mathcal{C})$ is a quasi-isomorphism if it induces isomorphisms on the cohomology groups H^n for every $n\in\mathbb{Z}$

Quasi-isomorphisms may not have inverses, for example

$$\begin{tikzcd}
X^{\cdot} \arrow[d] & \cdots \underset{(0)}{0} \arrow[r] & \underset{(1)}{\mathbb{Z}} \arrow[r, "{[2]}"] \arrow[d] & \underset{(2)}{\mathbb{Z}} \arrow[r] \arrow[d] & 0 \cdots \\
Y^{\cdot} & \cdots 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "{[2]}"] & \mathbb{Z}/4\mathbb{Z} \arrow[r] & 0 \cdots
\end{tikzcd}$$

where $\mathbb{Z}/2\mathbb{Z} \xrightarrow{[2]} \mathbb{Z}/4\mathbb{Z}$

$0 \longmapsto 0$

$1 \longmapsto 2$

is a quasi-isomorphism but has no inverse

It is also true that not every collection of isomorphisms $H^i(X^{\cdot}) \xrightarrow{\sim} H^i(Y)$ comes from a quasi-isomorphism.

Snake lemma: consider a diagram

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

with exact rows. Then there exists a morphism $\ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha)$ fitting into an exact sequence

$$\begin{tikzcd}
\ker \alpha \arrow[r] & \ker \beta \arrow[r] & \ker \gamma \arrow[dll, out=0, in=180] \\
\operatorname{coker} \alpha \arrow[r] & \operatorname{coker} \beta \arrow[r] & \operatorname{coker} \gamma
\end{tikzcd}$$

and δ has a certain natural property (if we have two such diagrams with morphism $\square \to \square$ then the existing morphisms + δ's make a commutative diagram)

Proof for R-Mod (sketch): let $c \in C$ such that $\gamma(c) = 0$. Then c is the image of some $b \in B$ and $\beta(b)$ maps to 0 in C'. So $\beta(b) \in B' \in \operatorname{Im}(A' \to B')$. Changing the choice of b changes the element of A' by an element coming from A, so the image in $\operatorname{coker}(\alpha)$ is well-defined. Weibel uses a [illegible]

□

For general abelian categories one can:

- give a purely categorical proof (by Bergman)
- use the "Freyd-Mitchell embedding": any small abelian category embeds in R-Mod for some ring R
- consider the homs from an arbitrary element X to abelian groups.

Corollary: let

$$0 \to A^\cdot \to B^\cdot \to C^\cdot \to 0$$

be a short exact sequence in $Ch^\cdot(\mathcal{C})$. Then there exist maps

$$\partial^i : H^i(C^\cdot) \to H^{i+1}(A^\cdot) \qquad \forall i \in \mathbb{Z}$$

fitting into a long exact sequence

$$\to H^i(A^\cdot) \to H^i(B^\cdot) \to H^i(C^\cdot) \xrightarrow{\partial^i} H^{i+1}(A^\cdot) \to H^{i+1}(B^\cdot) \to H^{i+1}(C^\cdot) \to \ldots$$

Proof: we have exact sequences

$$0 \to \ker(d_A^n) \to \ker(d_B^n) \to \ker(d_C^n)$$

$$\operatorname{coker}(d_A^n) \to \operatorname{coker}(d_B^n) \to \operatorname{coker}(d_C^n) \to 0 \qquad \forall n \in \mathbb{Z}$$

so by snake lemma we deduce the existence of a diagram

$$\begin{tikzcd}
H^n(A) \arrow[r] \arrow[d] & H^n(B) \arrow[r] \arrow[d] & H^n(C) \arrow[d] \arrow[ddll, out=0, in=180, "\partial^n"] & \\
\operatorname{coker} d_A^{n-1} \arrow[r] \arrow[d] & \operatorname{coker} d_B^{n-1} \arrow[r] \arrow[d] & \operatorname{coker} d_C^{n-1} \arrow[r] \arrow[d] & 0 \\
\ker d_A^{n+1} \arrow[r] \arrow[d] & \ker d_B^{n+1} \arrow[r] \arrow[d] & \ker d_C^{n+1} \arrow[d] & \\
H^{n+1}(A) \arrow[r] \arrow[d] & H^{n+1}(B) \arrow[r] & H^{n+1}(C) & \\
0 & & &
\end{tikzcd}$$

□

The snake arrow is our ∂^n and naturality of the snake lemma gives a natural property of ∂^i

Indeed, if we have

$$\begin{tikzcd}
0 \arrow[r] & A^{\bullet} \arrow[r] \arrow[d] & B^{\bullet} \arrow[r] \arrow[d] & C^{\bullet} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'^{\bullet} \arrow[r] & B'^{\bullet} \arrow[r] & C'^{\bullet} \arrow[r] & 0
\end{tikzcd}$$

then

$$\begin{tikzcd}
\cdots & H^n(C) \arrow[r, "\partial^n"] \arrow[d] & H^{n+1}(A) & \cdots \\
\cdots & H^n(C') \arrow[r, "\partial'^n"] & H^{n+1}(A') \arrow[from=u] & \cdots
\end{tikzcd}$$

Resolutions

Def: if $X \in Ob(\mathcal{C})$ we define a complex $[X] \in Ch^{\bullet}(\mathcal{C})$ that is X in degree 0 and 0 elsewhere. It is formally a functor $\mathcal{C} \longrightarrow Ch^{\bullet}(\mathcal{C})$.

A resolution of X is any quasi-isomorphism between $[X]$ and some object in $Ch^{\bullet}(\mathcal{C})$.