HOMOLOGICAL ALGEBRA

References: Weibel, Intro to homological algebra (uses homological convention rather than cohomological ones)
Gelfand-Manin, beautiful account but many typos   Rotman, rings & modules, Tor, Ext, etc.

Def: a category $\mathcal{C}$ consists of
- a collection of objects $Ob(\mathcal{C})$
- for every $X, Y \in Ob(\mathcal{C})$ a set $Hom_{\mathcal{C}}(X,Y)$ of morphisms from X to Y.
- $\forall X, Y, Z \in Ob(\mathcal{C})$ a map
$$Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \longrightarrow Hom_{\mathcal{C}}(X,Z)$$
called composition.

Moreover composition must satisfy some axioms:

- $Hom_{\mathcal{C}}(X,Y)$ and $Hom_{\mathcal{C}}(X',Y')$ are disjoint unless $X=X'$ and $Y=Y'$
- every object X has an identity morphism $id_X \in Hom_{\mathcal{C}}(X,X)$ having the property
$$id_Y \circ f = f \circ id_X = f \quad \forall f \in Hom_{\mathcal{C}}(X,Y)$$
- composition of morphisms is associative

EXAMPLES
- Grp objects = groups, morphisms = group hom.
- Top objects = topological spaces, morphisms = continuous maps
- Set, Ab Grp, ...
- any group forms a category with one object + the group elements as morphisms

Remark: $Ob(\mathcal{C})$ is not required to be a set of objects – indeed there is no set of all sets. A category whose objects form a set is called small. We are assuming that morphisms between objects are a set (locally small category).

Def: a (covariant) functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ between two categories $\mathcal{C}, \mathcal{D}$ is the datum of
- a mapping
$$Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{D})$$
$$X \longmapsto F(X)$$
- for every $X, Y \in Ob(\mathcal{C})$ a map
$$Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(FX, FY)$$
$$\varphi \longmapsto F(\varphi)$$
respecting composition of morphisms and sending $F(id_X) = id_{FX}$

EXAMPLES
- Forgetful functors
$$\text{Grp} \longrightarrow \text{Set}$$
$$G \longmapsto \text{set of elements of } G$$
$$\varphi \longmapsto \varphi$$
$$\text{Top} \longrightarrow \text{Set}$$
$$X \longmapsto \text{set of elements of } X$$
$$\varphi \longmapsto \varphi$$
$$\text{AbGrp} \longrightarrow \text{Grp}$$
- Homology (singular, cellular, simplicial, ...) $\text{Top} \to \text{Ab Grp}$ (with $\mathbb{Z}$ coefficients)

• "Free" functors $\quad Set \longrightarrow Vect_{\mathbb{C}}$

$\quad S \longmapsto$ vector space on S as a basis

Def: If $\mathcal{C}$ is a category then $\mathcal{C}^{op}$ is a category with

$$Ob(\mathcal{C}^{op}) = Ob(\mathcal{C}) \qquad Hom_{\mathcal{C}^{op}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$$

A contravariant functor $\mathcal{C} \longrightarrow \mathcal{D}$ is a covariant functor $\mathcal{C}^{op} \longrightarrow \mathcal{D}$

EXAMPLES

- $\mathcal{C} \longrightarrow \mathcal{C}^{op}$ (identity on objects) is a contravariant functor
- duals of vector spaces $\quad Vect_k \longrightarrow Vect_k$, $\quad V \longmapsto V^*$
- cohomology functors

Remark: If you try to form Cat the category of categories with functors as morphisms then it won't be locally small.

Def: Let $\mathcal{C}, \mathcal{D}$ be categories and F, G covariant or contravariant functors $\mathcal{C} \longrightarrow \mathcal{D}$. A natural transformation $F \xrightarrow{\varphi} G$ consists of a collection of morphisms:

$$\forall x \in Ob(\mathcal{C}) \quad \varphi_x \in Hom_{\mathcal{D}}(Fx, Gx)$$

such that for every morphism $\alpha : X \longrightarrow Y$ in $\mathcal{C}$ the diagram is commutative.

$$\begin{tikzcd} F(X) \arrow[r, "F(\alpha)"] \arrow[d, "\varphi_X"'] & F(Y) \arrow[d, "\varphi_Y"] \\ G(X) \arrow[r, "G(\alpha)"'] & G(Y) \end{tikzcd}$$

The idea is that F and G should do related things to related objects.

EXAMPLE

- In the category of groups Grp the functor $F : Grp \longrightarrow Grp$, $G \longmapsto G^{op}$ $\quad$ G^{op} is the group G with reversed operation

$G^{op} = (G, *) \quad g * h = hg$

There is a natural transformation $\begin{pmatrix}\text{identity}\\ \text{functor}\end{pmatrix} \longrightarrow F$ given by sending $G \in Ob(Grp)$ to the morphism $G \longrightarrow G^{op}$, $g \longmapsto g^{-1}$. Commutativity of the diagram just says that [illegible]

We can also [illegible]

- For any topological space X and base point $x \in X$ we can form $\pi_1(X,x)$ and $H_1(X,\mathbb{Z})$. It is a fact that $H_1(X,\mathbb{Z})$ is the abelianization of $\pi_1(X,x)$:

$$\pi_1(X,x) \big/ \text{commutator subgroup} \xrightarrow{\simeq} H_1(X,\mathbb{Z})$$

is a natural transformation of functors $Top' \longrightarrow AbGrp$ $\qquad Top'$ = pointed spaces.

- Let $F : \cancel{Ab}\ Set \longrightarrow AbGrp$ be the free abelian group functor and $G : AbGrp \longrightarrow Set$ the forgetful one.

$$F \circ G : AbGrp \longrightarrow AbGrp \qquad G \circ F : Set \longrightarrow Set$$

are not the identity functors, but still there is some structure.

There are natural transformations $F \circ G \to id_{AbGrp}$ and $id_{Set} \to G \circ F$
$G \circ F \to$

Adjoint functors

In the example above we have a bit more structure. If S is any set and T any abelian group then

$$(\text{maps of set from } S \text{ to } T) = (\text{maps of abelian groups } F(S) \to T)$$

because any map of sets $S \to T$ can be uniquely extended to a group morphism $F(S) \to T$. These are natural:

$$\mathrm{Hom}_{Set}(S, G(T)) \xrightarrow{\simeq} \mathrm{Hom}_{AbGrp}(F(S), T)$$

agree as functors

$$\begin{aligned} Set^{op} \times AbGrp &\longrightarrow Set \\ (S,T) &\longmapsto \mathrm{Hom}_{Set}(S, GT) \\ &\longmapsto \mathrm{Hom}_{Grp}(FS, T) \end{aligned}$$

this structure is called an adjunction

Let $\mathcal{C}$, $\mathcal{D}$ be categories and [illegible] functors [illegible]

$$\begin{aligned} \mathcal{C}^{op} \times \mathcal{D} &\to Set \\ (X,Y) &\mapsto \mathrm{Hom}_{\mathcal{D}}(FX, Y) \\ &\mapsto \mathrm{Hom}_{\mathcal{C}}(X, GY) \end{aligned}$$

are naturally iso [illegible]

We say that F is [illegible] adjoint to G and G [illegible] adjoint to F

Exercise: 1) Show that the forgetful functor $Top \to Set$ has a left adjoint and describe it explicitly.
2) Does it have a right adjoint?

Def: an additive category is a category $\mathcal{C}$ together with a binary operation $+$ on every Hom-set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ making it into an abelian group for every $X, Y \in Ob(\mathcal{C})$ such that the following axioms are satisfied

- $\exists\, 0 \in Ob(\mathcal{C})$ zero object such that $\mathrm{Hom}_{\mathcal{C}}(0, X)$ - trivial abelian group $\forall X$
- composition of morphisms is bilinear
- any two objects $X, Y \in Ob(\mathcal{C})$ have a direct sum $X \oplus Y$ satisfying the universal property.