TCC Homological Algebra: Assignment #1

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This is the first of 4 problem sheets. Solutions should be submitted to me (via any appropriate method[1]) by **noon on 2nd November**. This problem sheet will be marked out of a total of 25; the number of marks available for each question is indicated.

Note that rings are assumed to be unital (i.e. having a multiplicative identity element 1), and ring homomorphisms are assumed to map 1 to 1.

Review on categories, functors, and natural transformations

1. We saw in lectures that any group G gives rise to a category (with one object). Let $\underline{G}$ be this category.
   (a) [1 point] Show that a group homomorphism $G \to H$ is the same thing as a functor $\underline{G} \to \underline{H}$.
   (b) [2 points] Given two group homomorphisms $G \to H$, when does there exist a natural transformation between the corresponding functors?

2. [3 points] Let **Grp** be the category of groups and group homomorphisms. Find two distinct functors $F : \mathbf{Grp} \to \mathbf{Grp}$ with $F(G) = G$ for every group G.

Adjunctions

3. For G a group, its *group ring* $\mathbf{Z}[G]$ is the ring whose elements are finite formal sums $\sum_{i=1}^{N} a_i[g_i]$ with $a_i \in \mathbf{Z}$ and $g_i \in G$; addition is defined in the obvious fashion, and multiplication by $[g] \cdot [h] = [gh]$.
   (a) [2 points] Check, carefully, that there exists a functor $F : \mathbf{Grp} \to \mathbf{Ring}$ which acts on objects as $G \mapsto \mathbf{Z}[G]$, and which sends a morphism $\phi : G \to H$ to $F(\phi) : \sum a_i[g_i] \mapsto \sum a_i[\phi(g_i)]$.
   (b) [2 points] Construct a functor **Ring** $\to$ **Grp** which is right adjoint to F. (*You should define the necessary bijections between hom-sets, but you need not prove that they are natural.*)

4. [2 points] Let F be the forgetful functor **Top** $\to$ **Set**, where **Top** is the category of topological spaces and continuous maps. Show that F has both a left and a right adjoint, and describe them. Are they equal?

Additive categories, kernels, and cokernels

5. [2 points] Let C be a category. Suppose that C satisfies axioms (A2) and (A3) of an additive category, and we have *two* binary operations $\boxplus$ and $\boxdot$ on every hom-set in C, both of which satisfy the axiom (A1).
   (a) Show that for any two objects X_1, X_2, the maps $i_j : X_j \to X_1 \oplus X_2$ and $p_j : X_1 \oplus X_2 \to X_j$ ($j = 1, 2$) appearing in the definition of $X_1 \oplus X_2$ must satisfy

$$i_1 \circ p_1 \boxplus i_2 \circ p_2 = i_1 \circ p_1 \boxdot i_2 \circ p_2 = \mathrm{id}_{X_1 \oplus X_2}.$$

[1] For non-Warwick students, the simplest method is probably to scan your work and email it to me; but it is *your* responsibility to ensure that the scanned files are legible!

(b) Hence show that $\boxdot$ and $\boxplus$ are identical.

6. [1 point] Verify that the usual kernel of a morphism in **AbGrp** satisfies the definition of a category-theoretic kernel.

7. [2 points] Let $\mathbf{Ban}_{\mathbf{C}}$ be the category of complex Banach spaces and bounded linear maps. Do cokernels exist in $\mathbf{Ban}_{\mathbf{C}}$?

8. [3 points] Let $\mathcal{C}$ be the category whose objects are *torsion-free* abelian groups, with group homomorphisms as morphisms. Show that kernels and cokernels exist in $\mathcal{C}$, and give an example to show that the cokernel of a morphism in $\mathcal{C}$ is not always the same as its cokernel in **AbGrp**.

9. [3 points] Find an additive category in which not every morphism has a kernel.

10. [2 points] Let C be an additive category in which all kernels and cokernels exist. Show that for every morphism $\phi : X \to Y$ in C, there is a unique morphism

$$\mathrm{Coim}(\phi) \to \mathrm{Im}(\phi)$$

such that the composite $X \to \mathrm{Coim}(\phi) \to \mathrm{Im}(\phi) \to Y$ is ϕ.

Give an example to show that this may not be an isomorphism.