TCC Homological Algebra: Assignment #2 (Solutions)

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6/12/19

Note that rings are not necessarily commutative, but are always assumed to be unital (i.e. having a multiplicative identity element 1), and ring homomorphisms are assumed to map 1 to 1. The notation Ab denotes the category of abelian groups, and R-Mod the category of left modules over the ring R.

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1. Let FAb denote the full subcategory of Ab whose objects are finite abelian groups.

   (a) [1 point] Show that FAb is an abelian category. (You may assume that Ab is abelian.)

   Solution: If f is a morphism in FAb, then it has a kernel and cokernel in Ab, and these are also objects of FAb, since a subgroup or quotient of a finite group is finite. Since they satisfy a universal property in the larger category Ab, they also satisfy the same universal property in FAb. Thus kernels and cokernels exist for all morphisms in FAb and they coincide with kernels and cokernels in Ab. Moreover, the arrow

   $$\overline{f} : \mathrm{coker}(\ker f) \to \ker(\mathrm{coker}\, f)$$

   is an isomorphism in Ab and hence also in FAb. Thus FAb is abelian.

   (b) [2 points] Show that the only injective object in FAb is 0. (Hint: if G is a non-zero injective, consider homomorphisms from cyclic groups to G.)

   Solution: Let G be a non-zero injective object. Since G is finite, there is a maximal $n > 1$ such that G has an element of order n. Let g_n be an element of order n. Then there cannot exist $h \in G$ such that $n \cdot h = g_n$, since this h would necessarily have order $n^2 > n$. Thus the homomorphism $C_n \to G$ sending the generator to g_n cannot be extended to a homomorphism $C_{n^2} \to G$, contradicting the injectivity of G.

2. [3 points] Let $\mathcal{C}$ be an abelian category, Σ a set, and for each $\sigma \in \Sigma$, let M_σ be an object of $\mathcal{C}$. We define $\prod_{\sigma\in\Sigma} M_\sigma$ to be the limit of the diagram consisting of the objects M_σ with no morphisms between them, and $\bigoplus_{\sigma\in\Sigma} M_\sigma$ its colimit, assuming these limits exist.

   (a) Show that if M_σ is projective for all σ, then so is $\bigoplus_{\sigma\in\Sigma} M_\sigma$.

   Solution: Let $M = \bigoplus_{\sigma\in\Sigma} M_\sigma$, let $f : M \to Y$ be any morphism, and $p : X \to Y$ an epimorphism. Since M is a colimit, there are canonical maps $i_\sigma : M_\sigma \to M$ for all σ. Let $f_\sigma = f \circ i_\sigma : M_\sigma \to Y$. Since M_σ is projective this lifts to a map $\tilde{f}_\sigma : M_\sigma \to X$. By the universal property of M as colimit, the maps $\tilde{f}_\sigma$ assemble into a map $\tilde{f} : M \to X$ which lifts f. So M is projective.

   (b) Show that if M_σ is injective for all σ, then so is $\prod_{\sigma\in\Sigma} M_\sigma$.

   Solution: Apply (a) to the corresponding diagram in the opposite category.

(c) Show that $\mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_{\sigma\in\Sigma} M_\sigma, Z\right) = \prod_{\sigma\in\Sigma} \mathrm{Hom}_{\mathcal{C}}(M_\sigma, Z)$ for any object Z of $\mathcal{C}$.

> **Solution:** Let $M = \bigoplus_{\sigma\in\Sigma} M_\sigma$ as before, and $i_\sigma : M_\sigma \to M$. Then mapping $f : M \to Z$ to $(i_\sigma \circ f)_{\sigma\in\Sigma}$ defines a map $\mathrm{Hom}(M, Z) \to \prod_\sigma \mathrm{Hom}(M_\sigma, Z)$, and the universal property of the colimit asserts that this map is a bijection.

3. [3 points] Let $\mathcal{C}$ be an abelian category and $A^\bullet, B^\bullet$ cochain complexes over $\mathcal{C}$. Define a complex $\mathcal{H} = \underline{Hom}(A^\bullet, B^\bullet) \in \mathrm{Ch}(\underline{\mathrm{Ab}})$ by $\mathcal{H}^i = \prod_{j\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{C}}(A^j, B^{j+i})$.

(a) Show that the maps $d^i_{\mathcal{H}} : \mathcal{H}^i \to \mathcal{H}^{i+1}$ defined by

$$d^i_{\mathcal{H}}\left((f^j)_{j\in\mathbf{Z}}\right) = (f^{j+1}\circ d^j_A - (-1)^i d^{j+i}_B \circ f^j)_{j\in\mathbf{Z}}$$

are well-defined, and satisfy $d^{i+1}_{\mathcal{H}} \circ d^i_{\mathcal{H}} = 0$.

> **Solution:** Let f denote the element $(f_j)_{j\in\mathbf{Z}}$. Clearly both $f^{j+1}\circ d^j_A$ and $d^{j+i}_B \circ f^j$ are homomorphisms $A^j \to B^{j+i+1}$, so $d^i_{\mathcal{H}}(f)$ is a collection of morphisms with the correct sources and targets.
> Let us write $g = d^i_{\mathcal{H}}(f)$. Then we have
>
> $$\begin{aligned} d^{i+1}_{\mathcal{H}}(g) &= g^{j+1}\circ d^j_A + (-1)^i d^{i+j+1}_B \circ g^j \\ &= \left(f^{j+2}\circ d^{j+1}_A - (-1)^i d^{i+j+1}_B \circ f^{j+1}\right)\circ d^j_A + (-1)^i d^{i+j+1}_B \circ \left(f^{j+1}\circ d^j_A - (-1)^i d^{j+i}_B \circ f^j\right) \\ &= \Big(f^{j+2}\circ \underbrace{d^{j+1}_A \circ d^j_A}_{=0}\Big) - (-1)^i \left(d^{i+j+1}_B \circ f^{j+1}\circ d^j_A\right) \\ &\quad + (-1)^i \left(d^{i+j+1}_B \circ f^{j+1}\circ d^j_A\right) - \Big(\underbrace{d^{i+j+1}_B \circ d^{j+i}_B}_{=0} \circ f^j\Big) \\ &= 0. \end{aligned}$$

(b) Show that $\ker(d^0_{\mathcal{H}}) = \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(A^\bullet, B^\bullet)$.

> **Solution:** An element f of $\mathcal{H}^0$ is a collection of maps $f^j : A^j \to B^j$. It satisfies $d^0_{\mathcal{H}}(f) = 0$ iff $f^{j+1}\circ d^j_A = d^{j+1}_B \circ f^j$ for all j, which is precisely the definition of a cochain map.

(c) Show that $\mathrm{im}(d^{(-1)}_{\mathcal{H}})$ is the null-homotopic maps.

> **Solution:** An element f of $\mathcal{H}^0$ is null-homotopic if and only if there exist maps $s^j : A^j \to B^{j-1}$ such that $f^j = s_j \circ d^j_A + d^{j-1}_B \circ s^j$. This is precisely the assertion that $f = d^{(-1)}_{\mathcal{H}}((s^j)_{j\in\mathbf{Z}})$.

4. [2 points] Let X, Y be two objects in an abelian category $\mathcal{C}$, and $I^\bullet$, $J^\bullet$ injective resolutions of X, Y respectively. Let $f^\bullet : I^\bullet \to J^\bullet$ a morphism of complexes which induces the zero map $X \to Y$ on H^0. Show that $f^\bullet$ is null-homotopic.

[Hint: We are looking for maps $s^i : I^i \to J^{i-1}$ for all i such that $f = ds + sd$. For $i \le 0$ the target of s^i is the zero object, so the first nontrivial step is to construct $s^1 : I^1 \to J^0$ compatible with f^0. Then look for an opportunity to induct on i.]

> **Solution:** Since the map $f^0 : I^0 \to J^0$ induces the zero map on X, it factors through I^0/X. The map d_I^0 induces a monomorphism $I^0/X \to I^1$, so by injectivity of J^0, we can find s^1 such that $f^0 = s^1 \circ d_I^0$.
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> Now let $n \geq 1$, and let us suppose we have constructed morphisms $s^i : I^i \to J^{i-1}$, for $1 \leq i \leq n$, such that the identity $f^i = d_J^{i-1} \circ s^i + s^{i+1} \circ d_I^i$ holds for $1 \leq i \leq n-1$ (with $s^0 = 0$). Then $h^n := f^n - d^{n-1} \circ s^n$ is a homomorphism $I^n \to J^n$ which satisfies
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> $$h^n \circ d_I^{n-1} = (f - ds) \circ d = fd - dsd = fd - d(f - ds) = (fd - df) + dds = 0.$$
>
> (dropping indices for clarity). So h^n factors through $I^n / \operatorname{Im}(d_I^{n-1})$, which is a subobject of I^{n+1}. Via the injectivity of J^n, we can define $s^{n+1} : I^{n+1} \to J^n$ such that $h^n = s^{n+1} \circ d$. So by induction we can find s^n for all $n \geq 1$ satisfying the required identity.
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> *[Note that we do not need to use the assumption that $I^\bullet$ has injective terms or that $J^\bullet$ is exact.]*

5. [2 points] Give an example of a morphism in $\mathrm{Ch}(\underline{\mathrm{Ab}})$ which is a quasi-isomorphism, but not a homotopy equivalence.

> **Solution:** Let $A = [\mathbf{Z}]$ and $B = [\mathbf{Q} \longrightarrow \mathbf{Q}/\mathbf{Z}]$, and let f be the natural map $A \to B$ given by the inclusion $\mathbf{Z} \hookrightarrow \mathbf{Q}$. We saw in lectures that f is a quasi-isomorphism.
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> However, we have $\mathrm{Hom}_{\underline{\mathrm{Ab}}}(\mathbf{Q}, \mathbf{Z}) = 0$, so $\mathrm{Hom}_{\mathrm{Ch}(\underline{\mathrm{Ab}})}(B, A) = 0$. So if f had a homotopy inverse, it would have to be the zero map; thus the zero map $A \to A$ would have to be homotopic to the identity and hence would have to induce the identity on $H^0(A) = \mathbf{Z}$. This is impossible since $\mathbf{Z}$ is not the zero ring. So f is not a homotopy equivalence.

6. [2 points] Show that if $F : \mathcal{C} \to \mathcal{D}$ a left-exact functor between abelian categories, and $0 \to A \to B \to C \to 0$ is an exact sequence with A injective, then $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact. *[Hint: We are **not** assuming that $\mathcal{C}$ has enough injectives, so it is not enough to say that $R^1(F)(A) = 0$.]*

> **Solution:** Since A is injective, the identity map $A \to A$ has to extend to a map $s : B \to A$. This gives a splitting of the exact sequence, so $B \cong A \oplus C$ with the maps $A \to B$ and $B \to C$ being the inclusion and projection maps. Since F is additive, it preserves direct sums, so $0 \to F(A) \to F(A \oplus C) \to F(C) \to 0$ is exact.

7. [1 point] Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be abelian categories and $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ left-exact functors. Assume $\mathcal{C}$ has enough injectives, G is exact, and F is left-exact. Show that $R^i(G \circ F) = G \circ R^i(F)$ for all i, as functors $\mathcal{C} \to \mathcal{E}$.

> **Solution:** By definition $R^i(G \circ F)(X) = H^i\left((G \circ F)(I^\bullet)\right)$ where $I^\bullet$ is an injective resolution of X. Since G is exact, it commutes with cohomology, so this is the same as $G\left(H^i(F(I^\bullet))\right) = G(R^i(F)(X))$.
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> *[If $\mathcal{D}$ has enough injectives then this is a trivial consequence of the Grothendieck spectral sequence, but this is not assumed.]*

8. [3 points] Let $G = C_2 = \{1, \sigma\}$.
   (a) Show that
   $$\dots \mathbf{Z}[G] \xrightarrow{\sigma - 1} \mathbf{Z}[G] \xrightarrow{\sigma + 1} \mathbf{Z}[G] \xrightarrow{\sigma - 1} \mathbf{Z}[G]$$
   is a projective resolution of the trivial module $\mathbf{Z}$ as a $\mathbf{Z}[G]$-module.

Solution: This is obviously a complex since $(\sigma+1)(\sigma-1) = \sigma^2 - 1 = 1 - 1 = 0$. So we must check it has the correct cohomology.

A generic element of $\mathbf{Z}[G]$ looks like $a + b\sigma$ for $a, b \in \mathbf{Z}$, and we have $(\sigma \pm 1)(a + b\sigma) = (a \pm b)(\sigma \pm 1)$. Thus the image of multiplication by $\sigma \pm 1$ is $\mathbf{Z} \cdot (\sigma \pm 1)$, and its kernel is $\{a + b\sigma : a \pm b = 0\} = \mathbf{Z} \cdot (\sigma \mp 1)$, which proves that the cohomology in all degrees $\neq 0$ is trivial. Meanwhile, the image of the last map is exactly the kernel of the surjection $\mathbf{Z}[G] \to \mathbf{Z}$, $a + b\sigma \mapsto a + b$.

(b) Hence compute the cohomology groups of

i. $\mathbf{Z}$ with the trivial G-action;

Solution: We must compute the cohomology of the complex of homomorphisms from the above resolution to $\mathbf{Z}$, which is

$$\mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \dots$$

So the cohomology is $\mathbf{Z}$ in degree 0, trivial in odd degrees, and $\mathbf{Z}/2\mathbf{Z}$ in positive even degrees.

ii. $\mathbf{Z}$ with the generator σ acting as -1.

Solution: Now we obtain the complex

$$\mathbf{Z} \xrightarrow{-2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{-2} \mathbf{Z} \dots$$

so the cohomology is 0 in even degrees (including degree 0) and $\mathbf{Z}/2\mathbf{Z}$ in odd degrees.

9. Let R be a ring, A, B objects of $\underline{R\text{-Mod}}$, and $\sigma \in \operatorname{Ext}^1(A, B)$, represented by a homomorphism $f \in \operatorname{Hom}(A, Z^1(I^\bullet))$ where $I^\bullet$ is an injective resolution of B.

(a) [1 point] Show that the module

$$E = \{(x, a) \in I^0 \oplus A : d(x) = f(a)\},$$

with the obvious maps from B and to A, defines an extension of A by B; and show that the equivalence class of this extension depends only on σ and not on the representative f.

Solution: It is clear that E is an R-module, and $(x, a) \mapsto a$ is an R-module homomorphism $E \to A$. The kernel of this homomorphism is $\{(x, 0) : x \in \ker(I^0 \to I^1)\}$, and since $\ker(I^0 \to I^1) = B$, this gives an exact sequence $0 \to B \to E \to A$. However, since the target of f is $Z^1(I^\bullet)$, and $I^\bullet$ is exact, for every $a \in A$ there is x such that $f(a) = d(x)$; thus $E \to A$ is also surjective, so E is an extension of A by B.

If we let $f' = f + d \circ h$, where h is a homomorphism $A \to I^0$, and E' denotes the extension corresponding to f', then $(x, a) \mapsto (x + h(a), a)$ is clearly an isomorphism $E \to E'$ compatible with the maps to A and from B, so $0 \to B \to E \to A \to 0$ and $0 \to B \to E' \to A \to 0$ are equivalent as extensions of A by B. Thus the equivalence class of the extension depends only on the element σ.