TCC Homological Algebra: Assignment #3 (Solutions)

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15/1/20

Note that rings are not necessarily commutative, but are always assumed to be unital (i.e. having a multiplicative identity element 1), and ring homomorphisms are assumed to map 1 to 1. The notation $\underline{\text{Ab}}$ denotes the category of abelian groups, and $\underline{R\text{-Mod}}$ the category of left modules over the ring R. If $\mathcal{C}$ is an abelian category, then $\mathrm{Ch}(\mathcal{C})$ denotes the category of cochain complexes over $\mathcal{C}$, and $\mathrm{Ch}^+(\mathcal{C})$ the full subcategory of bounded-below complexes.

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1. *(Borrowed from Pete Clark)* Let R be a commutative ring and M, N be R-modules.

   (a) [1 point] Show that the groups $\mathrm{Ext}^i_R(M, N)$ are also naturally R-modules.

   Solution: Since R is commutative, there is a natural R-module structure on $\mathrm{Hom}_R(M, N)$ for any R-modules M, N, given by $(r \cdot \phi)(m) = \phi(rm) = r\phi(m)$. This gives a functor $\mathrm{Hom} : \underline{R\text{-Mod}} \times \underline{R\text{-Mod}} \to \underline{R\text{-Mod}}$ from which the usual Hom functor is obtained by composing with the forgetful functor $\underline{R\text{-Mod}} \to \underline{\text{Ab}}$.

   We have defined $\mathrm{Ext}^i_R(M, N)$ as the i-th homology of the complex $\mathrm{Hom}_R(M, I^\bullet)$, where $I^\bullet$ is an injective resolution of N in $\underline{R\text{-Mod}}$. By the above, this complex naturally lives in $\mathrm{Ch}(\underline{R\text{-Mod}})$, and the forgetful functor commutes with taking homology. So the homology (in $\underline{\text{Ab}}$) of $\mathrm{Hom}_R(M, I^\bullet)$ is also naturally an R-module.

   (b) [2 points] Let $r \in R$ and let $\mu : N \to N$ be the map $x \mapsto rx$. Show that for any i, the map $\mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N)$ induced by μ via the functoriality of $\mathrm{Ext}^i(M, -)$ is also multiplication by r. Show a similar result for the multiplication-by-r map $M \to M$.

   Solution: By definition, the map $\mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N)$ induced by μ is the i-th homology of the map of complexes $\mathrm{Hom}_R(M, I^\bullet) \to \mathrm{Hom}_R(M, I^\bullet)$ given by composing a homomorphism with $\tilde{\mu}$, where $\tilde{\mu}$ is a lifting of μ to a map of complexes $I^\bullet \to I^\bullet$. However, one valid choice of $\tilde{\mu}$ is the map given by multiplication by r on each I^j, which is exactly the R-module structure on Ext^i defined above (using the formula $(r \cdot \phi)(x) = r\phi(x)$)

   The second statement is rather simpler: if ν denotes the multiplication-by-R map on M *[apologies for the notation!]* then the map $\mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N)$ induced by ν is given by pre-composing homomorphisms with ν; using the other formula $(r \cdot \phi)(x) = \phi(rx)$, this again recovers the R-module structure of $\mathrm{Ext}^i_R(M, N)$.

2. Let G be a group and $H \trianglelefteq G$ a subgroup isomorphic to $(\mathbf{Z}, +)$.

   (a) [1 point] Show that for any G-module M, we have $H^i(H, M) = 0$ for $i \notin \{0, 1\}$.

   Solution: Let h be a generator of H. Then $\mathbf{Z}[H] \cong \mathbf{Z}[X, X^{-1}]$, by mapping h to X. Since $\mathbf{Z}[X, X^{-1}]$ is an integral domain, multiplication by $X - 1$ is injective as a map $\mathbf{Z}[X, X^{-1}] \to \mathbf{Z}[X, X^{-1}]$, and its cokernel is $\mathbf{Z}$. So the complex

   $$\left[\mathbf{Z}[H] \xrightarrow{h-1} \mathbf{Z}[H]\right]$$

is a projective resolution of $\mathbf{Z}$ in $\mathbf{Z}[H]$-Mod, and thus for any H-module M, the cohomology $H^*(H, M)$ is computed by the complex

$$M \xrightarrow{h-1} M$$

which is nontrivial only in degrees 0 and 1.

(b) [1 point] Show that there is a long exact sequence

$$\dots \to H^n(G/H, H^0(H, M)) \to H^n(G, M) \to H^{n-1}(G/H, H^1(H, M)) \to H^{n+1}(G/H, H^0(H, M)) \to \dots$$

Solution: Applying Hochschild–Serre to G and H, we find that there is a spectral sequence with E_2 terms $E_2^{pq} = H^p(G/H, H^q(H, M))$ converging to $H^{p+q}(G, M)$.

By part (a), the E_2 page has only two non-zero rows, namely the $q = 0$ and $q = 1$ rows. So by a result from §5.6 of the lectures, the E_2 terms and the abutments $X^n = H^n(G, M)$ fit into a long exact sequence

$$\dots \to E_2^{(n,0)} \to X^n \to E_2^{(n-1,1)} \xrightarrow{d_2^{(n-1,1)}} E_2^{(n+1,0)} \to X^{n+1} \to \dots$$

as required.

[Several of you fell into the trap of thinking that this long exact sequence is actually the composition of a bunch of short exact sequences. This is false, in general, since there is no particular reason why the E_2 differentials should vanish. The "lots of short exact sequences" case would occur if we had $G/H \cong \mathbf{Z}$, not if $H \cong \mathbf{Z}$.]

3. [2 points] Let E be a (first-quadrant, cohomological) spectral sequence in Ab converging to $(X^n)_{n \geq 0}$, and suppose there is some r such that $E_r^{p,q}$ is finitely-generated for all (p, q) and zero for almost all (p, q). Show that X^n is finitely-generated for all n and zero for almost all n, and we have

$$\sum_{p,q} (-1)^{p+q} \operatorname{rank}\left(E_r^{p,q}\right) = \sum_n (-1)^n \operatorname{rank}(X^n).$$

Solution: Suppose that E_r^{pq} is fg for all (p, q) and zero for almost all (p, q) for some specific $r = r_0$. Since E_{r+1}^{pq} is a subquotient of E_r^{pq}, we see by induction that this holds for all $r \geq r_0$, and hence that E_∞^{pq} is fg for all (p, q) and zero for almost all. Hence, for every n, the group X^n has a filtration having finitely many graded pieces, each of which is fg, so it is itself fg. Moreover, each pair (p, q) contributes to X^n for a single n (namely $n = p + q$) so there are only finitely many n such that any graded piece of X^n is nonzero, so almost all X^n are zero.

Let us now evaluate the sums. We first note that if $A^\bullet$ is a bounded complex of finitely-generated abelian groups, then we have

$$\begin{aligned} \operatorname{rank}(A^n) &= \operatorname{rank}(\operatorname{im} d_A^{n-1}) + \operatorname{rank} H^n(A^\bullet) + \operatorname{rank}(A/\ker(d_A^n)) \\ &= \operatorname{rank}(\operatorname{im} d_A^{n-1}) + \operatorname{rank} H^n(A^\bullet) + \operatorname{rank}(\operatorname{im} d_A^n). \end{aligned}$$

Taking the alternating sum over i, the $\operatorname{im} d_A^n$ terms cancel out, and thus

$$\sum_i (-1)^i \operatorname{rank}(A^i) = \sum_i (-1)^i \operatorname{rank} H^i(A^\bullet).$$

We apply this to the complexes $A_r^\bullet$ given by $A_r^n = \bigoplus_{p+q=n} E_r^{pq}$, with differentials given by summing the differentials d_r^{pq} of the spectral sequence. Since $A_{r+1}^\bullet$ is the cohomology of $A_r^\bullet$, we deduce that the quantity $\chi_r := \sum_{p,q} (-1)^{p+q} \operatorname{rank} E_r^{pq}$ satisfies $\chi_{r+1} = \chi_r$ for all $r \ge r_0$. Thus $\chi_\infty = \chi_{r_0}$. However, since X^n has a filtration with graded pieces $\{E_\infty^{pq} : p+q = n\}$, we have

$$\operatorname{rank} X^n = \sum_{p+q=n} \operatorname{rank} E_\infty^{pq} \ \ \forall n$$

and hence

$$\sum_n (-1)^n \operatorname{rank} X^n = \sum_n \sum_{p+q=n} (-1)^{p+q} \operatorname{rank} E_\infty^{pq} = \chi_\infty = \chi_{r_0}$$

as required.

[*] Formulate and prove an analogous statement with "finitely-generated" replaced by "finite".

Solution: The generalisation I had in mind was the following: if there is an r such that E_r^{pq} is finite for all (p,q) and trivial for almost all (p,q), then X^n is finite for all n and trivial for almost all, and

$$\prod_{p,q} \left(\# E_r^{pq}\right)^{(-1)^{p+q}} = \prod_n (\# X^n)^{(-1)^n}.$$

(More generally still, one can formulate a version of this exercise in any abelian category that is "essentially small", i.e. isomorphism classes of objects form a set, using the idea of *Grothendieck groups*.)

4. [2 points] Let $G = \mathrm{SL}_2(k)$, where k is a finite field of characteristic $\ne 2$. Let M be k^2, with G acting via the standard left-multiplication action on column vectors. Show that $H^i(G, M) = 0$ for all i. *[Hint: Apply the Hochschild–Serre spectral sequence to $Z(G) \trianglelefteq G$.]*

Solution: The centre Z of G is ± 1, with the generator σ acting as -1 on G. From Sheet 2 we know that $H^i(Z, M)$ is computed by the complex

$$M \xrightarrow{\sigma - 1} M \xrightarrow{\sigma + 1} M \to \dots.$$

Since $\sigma + 1$ is the zero map map and $\sigma - 1$ is multiplication by -2, which is invertible in k, this complex is exact. Thus $H^i(Z, M)$ is zero for all i, and from the Hochschild–Serre exact sequence it follows that $H^i(G, M)$ is 0 for all i.

5. Let R be a ring and let $f : A^\bullet \to B^\bullet$ be a morphism in $\mathrm{Ch}(\underline{R\text{-Mod}})$. Recall the definition of the *mapping cone* $C_f^\bullet$ of f (with the corrected sign conventions given in Lecture 8).

   (a) [1 point] Show that $C_f^\bullet$ is a cochain complex, and the obvious projection and inclusion maps $g : C_f^\bullet \to A^\bullet[1]$ and $h : B^\bullet \to C_f^\bullet$ are cochain maps.

   Solution: Recall that we write [1] for the functor $\mathrm{Ch}(\mathcal{C}) \to \mathrm{Ch}(\mathcal{C})$ sending A to the complex $A[1]$ defined by $A[1]^i = A^{i+1}$, $d^i_{A[1]} = -d^i_A$; and with these conventions, $(C_f)^i = A^{i+1} \oplus B^i$, with the differential given by

   $$d^i_{C_f}((a,b)) = (-d_A^{i+1}(a), f^{i+1}(a) + d_B^i(b)).$$

We compute that

$$
\begin{aligned}
& d^{i+1}_{C_f}(d^i_{C_f}((a,b))) \\
&= d^{i+1}_{C_f}\big(-d^{i+1}_A(a), f^{i+1}(a)+d^i_B(b)\big) \\
&= \Big(-d^{i+2}_A(-d^{i+1}_A(a)), f^{i+2}(-d^{i+1}_A(a)) + d^{i+1}_B(f^{i+1}(a)+d^i_B(b))\Big) \\
&= \Big(0, (-f^{i+2}\circ d^{i+1}_A + d^{i+1}_B \circ f^{i+1})(a))\Big) = 0
\end{aligned}
$$

where the last equality follows from f being a cochain map. Thus $C^\bullet_f$ is also a complex. For g, h as above, we compute that both $d^i_{A[1]}\circ g^i$ and $g^{i+1}\circ d^i_{C^\bullet_f}$ send (a,b) to $-d^{i+1}_A(a)$, so g is a cochain map; similarly, both $h^{i+1}\circ d^i_B$ and $d^i_{C^\bullet_f}\circ h^i$ map b to $(0, d^i_B(b))$, so h is a cochain map.

(b) [2 points] Show that all three compositions $f\circ g$, $g\circ h$, and $h\circ f$ are null-homotopic.

Solution: *[This question was a little sloppily formulated, since $f\circ g$ doesn't quite make sense; it would have been more correct to write $f[1]\circ g$, where $f[1]$ is the morphism $A[1]\to B[1]$ given by $f[1]^i = f^{i+1}$.]*
One of these assertions is obvious: $g\circ h$ is the zero map, so it is certainly null-homotopic. So it suffices to prove the assertion for $f[1]\circ g$ and $h\circ f$.
Firstly, $h\circ f : A\to C_f$ is given by

$$a\in A^i \longmapsto f(a)\in B^i \longmapsto (0, f(a)) \in C^i_f.$$

Let $s: A^i \to C^{i-1}_f$ be given by $a\mapsto (a,0)$. Then we have $ds(a) = (-da, f(a))$ and $sd(a) = (da, 0)$. Hence $h\circ f = sd+ds$, so $h\circ f$ is null-homotopic.
Similarly, $f[1]\circ g : C_f\to B[1]$ is given by

$$(a,b)\in C^i_f \longmapsto a\in A^{i+1}\longmapsto f(a)\in B^{i+1}.$$

If we let $s: C^i_f \to B[1]^{i-1} = B^i$ be the map given by $(a,b)\mapsto b$, then $sd((a,b)) = db + f(a)$, while $ds((a,b)) = -db$ (the sign is because this is the differential of $A[1]$, not A). Thus $ds+sd = f\circ g$.

(c) [2 points] Show that if $g: A^\bullet\to B^\bullet$ is another morphism homotopic to f, then the complex $C^\bullet_g$ is homotopy-equivalent to $C^\bullet_f$, compatibly with the morphisms from $B^\bullet$ and to $A^\bullet[1]$.

Solution: Suppose that $s^i : A^i \to B^{i-1}$ are the components of the homotopy, so that $f-g = ds+sd$ (omitting unnecessary indices and brackets).
By definition both C^i_f and C^i_g are given by $A^{i+1}\oplus B^i$. We define a map $\lambda : C^i_f\to C^i_g$ by sending (a,b) to $(a, b+sa)$. Then we have

$$(d_{C_g}\circ\lambda)(a,b) = (-da, ga+db+dsa)$$

and

$$(\lambda\circ d_{C_f})(a,b) = \lambda(-da, fa+db) = (-da, fa+db-sda).$$

Since $f = g+ds+sd$ these are equal. Thus λ is a morphism of cochain complexes. Interchanging the role of f and g, and replacing s with $-s$, we obtain a map of complexes λ' in the other direction which is the inverse of λ. Thus C_f and C_g are isomorphic in Ch(R-Mod), and in particular are homotopy equivalent.

[This was a trick question, in some sense, because the natural argument actually proves something much stronger than I asked you for. But the weaker assertion that C_f and C_g are homotopic is enough to show that the mapping fibre is a well-defined operation on the homotopy category.]

6. Let $F : \mathcal{C} \to \mathcal{D}$ be a left-exact functor between abelian categories, with $\mathcal{C}$ having enough injectives. We defined the hyperderived functors $\mathbf{R}^i(F)$ as functors $\mathrm{Ch}^{\geqslant 0}(\mathcal{C}) \to \mathcal{D}$, where $\mathrm{Ch}^{\geqslant 0}(\mathcal{C})$ is the full subcategory of $\mathrm{Ch}^+(\mathcal{C})$ consisting of complexes that are zero in degrees < 0.
   (a) [1 point] Show that there is a unique extension of the functors $\mathbf{R}^i(F)$ to $\mathrm{Ch}^+(\mathcal{C})$ satsfying $\mathbf{R}^i(F)(X) = \mathbf{R}^0(F)(X[i])$.

Solution:
Lemma. Let $X \in \mathrm{Ch}^{\geqslant 0}(\mathcal{C})$. Then $X[-1] \in \mathrm{Ch}^{\geqslant 0}(\mathcal{C})$ as well, and we have $\mathbf{R}^i(F)(X[-1]) = \mathbf{R}^{i-1}(F)(X)$ (understood as 0 for $i = 0$).
Proof of Lemma. Let $I^{\bullet\bullet}$ be a Cartan-Eilenberg resolution of $X^\bullet$, and let $J^{\bullet\bullet}$ be the complex obtained by shifting this entire double complex one step to the right, and flipping the signs of all of the differentials. Then $J^{\bullet\bullet}$ is a CE resolution of $X[-1]$; and $\mathrm{Tot}\, F(J^{\bullet\bullet}) = (\mathrm{Tot}\, F(I^{\bullet\bullet}))\,[-1]$. Taking homology we deduce the lemma. □
Now, let $X \in \mathrm{Ch}^+(\mathcal{C})$. Then we have $X[-n] \in \mathrm{Ch}^{\geqslant 0}(\mathcal{C})$ for all sufficiently large n. If we define $R^i(F)(X) = R^{i+n}(F)(X[-n])$ for such an n, this is well-defined; and the lemma shows that this is independent of the choice of n.
Moreover, if n works for X, then $n + i$ works for $X[i]$, and we thus have

$$R^i(F)(X) := R^{n+i}(F)(X[-n]) = R^{n+i}(F)(X[i][-n-i]) =: R^0(F)(X[i]).$$

   (b) [1 point] Show that if $f : X^\bullet \to Y^\bullet$ is a quasi-isomorphism in $\mathrm{Ch}^{\geqslant 0}(\mathcal{C})$, then it induces isomorphisms $\mathbf{R}^i(F)(X) \to \mathbf{R}^i(F)(Y)$ for all Y.

Solution: We have spectral sequences $E_2^{pq} = (R^pF)(H^qX) \Rightarrow \mathbf{R}^{p+q}(F)(X)$ and similarly for Y. Since f is a quasi-isomorphism, it induces isomorphisms between the E_2 pages of these spectral sequences, and hence on the E_∞ pages as well. Thus the map $f : \mathbf{R}^i(F)(X) \to \mathbf{R}^i(F)(Y)$ is compatible with the filtrations induced by the spectral sequences, and induces isomorphisms on each graded piece, so it is an isomorphism.
*[**Alternatively**: It was mentioned in lecture 8 (during the discussion of derived categories) that $\mathbf{R}^i(F)(Y^\bullet)$ could be computed as the i-th homology of $F(I^\bullet)$, for any bounded-below complex of injectives $I^\bullet$ that is quasi-isomorphic to $Y^\bullet$ (and such complexes always exist). So one can simply take a quasi-iso $Y^\bullet \to I^\bullet$ with $I^\bullet$ injective, and compose it with f to get a quasi-iso $X^\bullet \to I^\bullet$, to see that the i-th homology of $F(I^\bullet)$ computes both $\mathbf{R}^i(F)(X)$ and $\mathbf{R}^i(F)(Y)$.*

However, if you use this argument, you should explain – with a proof or a reference to the notes – why $F(I^\bullet)$ computes the hyperderived functors.]

7. [4 points] *(Suggested by Sarah Zerbes)* Let $\mathcal{C}, \mathcal{D}$ be abelian categories with $\mathcal{C}$ having enough injectives, $F : \mathcal{C} \to \mathcal{D}$ a left-exact functor, and $f : A^\bullet \to B^\bullet$ a morphism of complexes in $\mathrm{Ch}^{\geqslant 0}(\mathcal{C})$. Let $C^\bullet = C_f^\bullet[-1]$, so we also have $C \in \mathrm{Ch}^{\geqslant 0}(\mathcal{C})$; this shifted mapping cone is sometimes known as the *mapping fibre*.

   Show that there is a spectral sequence in $\mathcal{D}$ converging to $\mathbf{R}^{p+q}(F)(C^\bullet)$, such that for each $q \geq 0$, the q-th row on the E_1 page, $E_1^{\bullet q}$, is the mapping fibre of the morphism $R^q(F)(f) : R^q(F)(A^\bullet) \to R^q(F)(B^\bullet)$ in $\mathrm{Ch}^{\geqslant 0}(\mathcal{D})$.

Solution: Let $I^{\bullet\bullet}$, $J^{\bullet\bullet}$ be Cartan–Eilenberg resolutions of $A^\bullet$, $B^\bullet$. Then f lifts to a map of double complexes $\tilde{f} : I^{\bullet\bullet} \to J^{\bullet\bullet}$.

Let $K^{\bullet\bullet}$ be the double complex with (p, q) term $I^{pq} \oplus J^{p-1,q}$, with vertical differentials being the direct sums of those of I and J, and with the horizontal differentials defined so that the q-th row is the mapping fibre of $I^{\bullet q} \to J^{\bullet q}$. Then $K^{\bullet\bullet}$ is a Cartan–Eilenberg resolution of $C^\bullet$.

By definition, $\mathbf{R}^n(F)(C)$ is the n-th cohomology of $\operatorname{Tot} F(K^{\bullet\bullet})$; so it is the abutment of two spectral sequences, corresponding to the "rows" and "columns" filtrations of the total complex. One of these has $E_1^{pq} = H_v^q(F(K^{p\bullet}))$. Since $K^{p\bullet}$ is the direct sum of $I^{p,\bullet}$ and $J^{p-1,\bullet}$ (and the vertical differentials respect this direct sum decomposition), we see that $E_1^{pq} = R^q(F)(A^p) \oplus R^{q-1}(F)(B^p)$, with horizontal differentials as claimed.