p-adic L-Functions of Modular Forms

by

Matthew Honnor

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Contents

1 Introduction

The p-adic numbers were first introduced by Kurt Hensel in the late $19^{\mathtt{th}}$ century. They are defined to be the completion of the rational numbers with respect to the p-adic absolute value. The immediate reason for studying them is due to Ostrowski's Theorem which states that every non-trivial absolute value on $\mathbb{Q}$ is one of $|\cdot|_p$ where p is either a prime number or ∞. Thus the p-adic numbers allow us to complete our picture of the completion of the rational numbers, using $\mathbb{R}$ and $\mathbb{Q}_p$. In this report, we will focus on analysis on the p-adic integers $\mathbb{Z}_p$; in particular section 2 of this report will be devoted to developing the theory of distributions of functions on $\mathbb{Z}_p$. This section will finish with a theorem which allows us to uniquely extend a distribution on locally polynomial functions of a small degree to r times differentiable functions. In section 3 of the report, we will change our focus to modular forms and after developing some of the theory we will be able to in section 4 use the work we have done on p-adic distributions to give an explicit distribution for r-times differentiable functions on $\mathbb{Z}_p$ This will be designed so that it takes interesting values for functions of the form x^n. In particular, these values will relate to the L-function attached to a modular form.

2 p-adic Analysis

2.1 p-adic Numbers

Throughout we shall let p denote a prime number and for $x \in \mathbb{R}$ let $[x]$ denote the integer part of x.

Definition 2.1. *Let $\mathbb{K}$ be a field. We define an absolute value to be a function $|\cdot| : \mathbb{K} \to \mathbb{R}_{\geq 0}$ that satisfies the following conditions:*

i) $|x| = 0$ *if and only if* $x = 0$*;*

ii) $|xy| = |x||y|$ *for all* $x, y \in \mathbb{K}$*;*

iii) $|x + y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{K}$*.*

We say that $|\cdot|$ is a non-archimedean absolute value if in addition it satisfies

iv) $|x + y| \leq \mathtt{max}\{|x|, |y|\}$ *for all* $x, y \in \mathbb{K}$*.*

If $|\cdot|$ does not satisfy iv) we call it an archimedean absolute value.

Definition 2.2. *We define* $\mathtt{ord}_p : \mathbb{Z} \to \mathbb{Z} \cup \{\infty\}$ *as follows,*

i) For non-zero $x \in \mathbb{Z}$, $\mathtt{ord}_p(x) = m$ where m is the largest integer for which $p^m | x$.

ii) $\mathtt{ord}_p(0) = \infty$*.*

We extend this definition to the rational numbers by defining $\mathtt{ord}_p(\frac{a}{b}) := \mathtt{ord}_p(a) - \mathtt{ord}_p(b)$*.*

Definition 2.3. *We define the p-adic absolute value* $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ *to be, for* $x \in \mathbb{Q}$, $|x|_p = p^{-\mathtt{ord}_p(x)}$.

Lemma 2.4. *We have the following properties for* $\mathtt{ord}_p$*:*

i) $\forall\, x, y \in \mathbb{Q}$ *we have* $\mathtt{ord}_p(xy) = \mathtt{ord}_p(x) + \mathtt{ord}_p(y)$*;*

ii) $\forall\, x, y \in \mathbb{Q}$ *we have* $\mathtt{ord}_p(x + y) \geq \mathtt{min}\{\mathtt{ord}_p(x), \mathtt{ord}_p(y)\}$.

Proof. Let $x, y \in \mathbb{Q}$ We write, $x = p^{m_x}\frac{a}{b}$ and $y = p^{m_y}\frac{c}{d}$ where $p \nmid abcd$. So $\mathtt{ord}_p(x) = m_x$ and $\mathtt{ord}_p(y) = m_y$.

Now consider $xy = p^{m_x+m_y}\frac{ac}{bd}$; since $p \nmid ac, bd$, we have $\mathtt{ord}_p(xy) = m_x + m_y$, so *i)* holds.

Now, without loss of generality suppose that $m_x \geq m_y$,

then $\mathtt{ord}_p(x + y) = p^{m_y}(p^{m_x-m_y}\frac{a}{b} + \frac{c}{d}) = p^{m_y}\frac{p^{m_x-m_y}ad+bc}{bd}$.

Since $p \nmid bd$, we have that $\mathtt{ord}_p(x+y) \geq m_y = \mathtt{min}\{\mathtt{ord}_p(x), \mathtt{ord}_p(y)\}$. Thus *ii)* holds. □

Proposition 2.5. *The p-adic absolute value is a non-achimedian absolute value.*

Proof. Let p be a prime number. We check the properties of an absolute value.

i) Let $x \in \mathbb{Q}$, then we have

$$|x|_p = 0 \iff p^{-\mathtt{ord}_p(x)} = 0 \iff \mathtt{ord}_p(x) = \infty \iff x = 0.$$

ii) Let $x, y \in \mathbb{Q}$, then

$$|xy|_p = p^{-\mathtt{ord}_p(xy)} = p^{-(\mathtt{ord}_p(x)+\mathtt{ord}_p(y))} = p^{-\mathtt{ord}_p(x)}p^{-\mathtt{ord}_p(y)} = |x|_p|y|_p.$$

iii) Let $x, y \in \mathbb{Q}$, then

$$|x + y|_p = p^{-\mathtt{ord}_p(x+y)} \leq p^{-\mathtt{min}\{\mathtt{ord}_p(x),\mathtt{ord}_p(y)\}} = \mathtt{max}\{|x|_p, |y|_p\}.$$

□

Definition 2.6. *Let* $\mathbb{K}$ *be a field and let* $|\cdot|$ *be an absolute value on* $\mathbb{K}$*. We say that* $\mathbb{K}$ *is complete with respect to* $|\cdot|$ *if every Cauchy sequence of elements of* $\mathbb{K}$ *has a limit in* $\mathbb{K}$.

Proposition 2.7. *Let* $\mathbb{K}$ *be a field with non-archimedean absolute value* $|\cdot|$*. Then a sequence* $(a_n)_{n=0}^{\infty}$ *in* $\mathbb{K}$ *is Cauchy if and only if* $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$.

Proof. First, suppose that $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence, then clearly $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$.

Now supposing the converse, let $m, n \in \mathbb{K}$ and without loss of generality suppose $m \geq n$. Then consider

$$|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - a_{m-2} + \cdots + a_{n-1} - a_n|.$$

Since $|\cdot|$ is a non-archimedean absolute value we have,

$$|a_m - a_n| \leq \mathtt{max}\{|a_m - a_{m-1}|, \ldots, |a_{n+1} - a_n|\}.$$

By our assumption, the above RHS tends to zero as $m, n \to \infty$ and thus $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence. □

Proposition 2.8. *The rational numbers are not complete with respect to the p-adic absolute value. (Example from [Gou03])*

Proof. First suppose that $p \neq 2$.

Choose $y \in \mathbb{Z}$ such that

i) y is not a square in $\mathbb{Q}$;

ii) $p \nmid a$;

iii) The congruence $X^2 \equiv y \pmod{p}$ has a solution.

To choose such a y, we first choose α be an integer not divisible by p, then choose $k \in \mathbb{Z}$ such that $p \nmid (\alpha^2 + kp)$. It is always possible to choose such a k as the difference between consecutive square numbers tends to infinity. We then set $y = \alpha^2 + kp$.

We now construct the sequence $(a_n)_{n=0}^{\infty}$ as follows.

Let a_0 be any solution of the congruence $X^2 \equiv y \pmod{p}$, which exists by *iii)* above.

We then define each a_n inductively such that $a_n \equiv a_{n-1} \pmod{p^n}$ and $a_n^2 \equiv y \pmod{p^{n+1}}$. We prove that such a_n exist by induction. We have the result for a_0, suppose the above holds for all n up to $m-1$.

Consider the congruence $\frac{a_{m-1}^2 - y}{p^m} + 2Xa_{m-1} \equiv 0 \pmod{p}$. $(\star)$

From our induction assumptions we have $\frac{a_{m-1}^2 - y}{p^m} + 2Xa_{m-1} \in \mathbb{Z}$ and $(2a_{m-1}, p) = 1$ (since $p \neq 2$ and $a_{m-1} \equiv a_0 \bmod p$). Thus the congruence has a solution, say z.

Now let $a_m = a_{m-1} + zp^m$, then it is clear that $a_m \equiv a_{m-1} \pmod{p^m}$.

Consider $a_m^2 = a_{m-1}^2 + 2zp^m a_{m-1} + z^2 p^m \equiv a_{m-1}^2 + 2zp^m a_{m-1} \pmod{p^{m+1}}$.

Since z satisfies $(\star)$, we have $a_{m-1}^2 + 2zp^m a_{m-1} \equiv y \pmod{p^{m+1}}$, so $a_n^2 \equiv y \pmod{p^{n+1}}$.

Thus we can define the sequence $(a_n)_{n=0}^{\infty}$ as stated above.

Now consider $|a_{n+1} - a_n|_p \leq p^{-(n+1)} \to 0$, by our construction. So the sequence $(a_n)_{n=0}^{\infty}$ is Cauchy.

But a_n tends to the square root of y, and we have chosen y to not be a square in $\mathbb{Q}$. Thus $\mathbb{Q}$ is not complete with respect to $|\cdot|_p$, with $p \neq 2$.

If $p = 2$ we define a sequence $(a_n)_{n=0}^{\infty}$, similarly to above but so that $a_n^3 \equiv 5 \pmod{p}$. □

It is worth noting that the method for generating the example above can be generalised; this result, Hensel's lemma, then allows for the construction of solutions to polynomials in the p-adic numbers (Definition 2.13).

In the same way that the real numbers are defined as the completion of the rational numbers with respect to the standard absolute value on $\mathbb{Q}$, we will define the p-adic numbers to be the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ as demonstrated below.

Definition 2.9. *Consider $\mathbb{Q}$ with the p-adic absolute value. We define the following sets.*

i) Let $\mathscr{C}_p(\mathbb{Q})$ be the set of Cauchy sequences in $\mathbb{Q}$ with respect to $|\cdot|_p$.

ii) Let $\mathscr{N}_p(\mathbb{Q}) \subset \mathscr{C}_p(\mathbb{Q})$ be the set of null sequences in $\mathbb{Q}$ with respect to $|\cdot|_p$.

We shall denote elements of $\mathscr{C}_p(\mathbb{Q})$ as (a_n) and define the following operations on $\mathscr{C}_p(\mathbb{Q})$.

i) $(a_n) + (b_n) = (a_n + b_n)$ for $a_n, b_n \in \mathscr{C}_p(\mathbb{Q})$.

ii) $(a_n)(b_n) = (a_n b_n)$ for $a_n, b_n \in \mathscr{C}_p(\mathbb{Q})$.

For $x \in \mathbb{Q}$, we shall denote its constant sequence as (x).

Lemma 2.10. *$\mathscr{C}_p(\mathbb{Q})$ is a ring, with operations defined above, additive identity (0) and multiplicative identity (1). In addition, $\mathscr{C}_p(\mathbb{Q})$ has maximal ideal $\mathscr{N}_p(\mathbb{Q})$.*

Proof. Checking that $\mathscr{C}_p(\mathbb{Q})$ satisfies the axioms for a ring is left as an exercise.

Let $(a_n) \in \mathscr{C}_p(\mathbb{Q})$, $(a_n) \notin \mathscr{N}_p(\mathbb{Q})$, let $I \subset \mathscr{C}_p(\mathbb{Q})$ be the ideal generated by (a_n) and $\mathscr{N}_p(\mathbb{Q})$. To prove that $\mathscr{N}_p(\mathbb{Q})$ is a maximal ideal in $\mathscr{C}_p(\mathbb{Q})$ we will show that I is equal to $\mathscr{C}_p(\mathbb{Q})$.

Since (a_n) does not tend to 0 and is a Cauchy sequence, there exists $c > 0$ and $N \in \mathbb{Z}_{\geq 0}$ such that $\forall\, n \geq N$, $|a_n|_p \geq c > 0$. In particular we have $a_n \neq 0$ for all $n \geq \mathbb{Z}_{\geq 0}$.

Now define a new sequence (b_n) by letting $b_n = 0$ if $n < N$ and $b_n = \frac{1}{a_n}$ if $n \geq N$.

Since, for $n \geq N$, $|b_{n+1} - b_n|_p = |\frac{1}{a_{n+1}} - \frac{1}{a_n}|_p \leq \frac{|a_{n+1}-a_n|_p}{|a_{n+1}a_n|_p} \leq \frac{|a_{n+1}-a_n|_p}{c^2} \to 0$, (b_n) is a Cauchy sequence i.e. $(b_n) \in \mathscr{C}_p(\mathbb{Q})$.

Now notice that $a_n b_n = \begin{cases} 0 \text{ if } n < N \\ 1 \text{ if } n \geq N. \end{cases}$

So the sequence $(a_n)(b_n)$ consists of a finite number of 0's followed by an infinite number of 1's.

Hence, $(1) - (a_n)(b_n) \in \mathscr{N}_p(\mathbb{Q})$.

Since I is an ideal, $(a_n)(b_n) \in I$. Recalling that I contains $\mathscr{N}_p(\mathbb{Q})$, we deduce that $(1) \in I$.

Thus, $I = \mathscr{C}_p(\mathbb{Q})$ as required. □

Lemma 2.11. *Let $(a_n) \in \mathscr{C}_p(\mathbb{Q})$, $(a_n) \notin \mathscr{N}_p(\mathbb{Q})$. Then the sequence of real numbers $|a_n|_p$ is eventually constant, i.e. $\exists\, N \in \mathbb{Z}_{\geq 0}$ such that $|a_n|_p = |a_m|_p$ for $n, m \geq N$.*

Proof. Since (a_n) is a Cauchy sequence which does not tend to 0, $\exists\, c > 0$ and $N_1 \in \mathbb{Z}_{\geq 0}$ such that $n \geq N_1 \implies |a_n|_p > c$.

Also $\exists\, N_2 \in \mathbb{Z}_{\geq 0}$ such that $m, n \geq N_2 \implies |a_n - a_m|_p < c$.

Set $N = \mathtt{max}\{N_1, N_2\}$. Now suppose $\exists\, n, m \geq N$ such that $|_p a_n|_p \neq |a_m|_p$, and without loss of generality let $|a_n|_p < |a_m|$.

Then, $n, m \geq N \implies |a_n - a_m|_p < c \leq |a_m|_p$.

Also, since $|\cdot|_p$ is a non-archimedean absolute values, we have $|a_m|_p \leq \mathtt{max}\{|a_m - a_n|_p, |a_n|_p\}$. Then by our assumption $|a_n|_p < |a_m|$ so $|a_m|_p \leq |a_n - a_m|_p$, which contradicts.

Thus $a_n = a_m$, $\forall\, n, m \geq N$. □

Lemma 2.12. *Let $(a_n) \in \mathscr{C}_p(\mathbb{Q})$, then*

i) the limit $\lim_{n\to\infty} |a_n|_p$ exists;

ii) if $(b_n) \in \mathscr{C}_p(\mathbb{Q})$ such that $(a_n) - (b_n) \in \mathscr{N}_p(\mathbb{Q})$, we have $\lim_{n\to\infty} |a_n|_p = \lim_{n\to\infty} |b_n|_p$.

Proof. Let $(a_n) \in \mathscr{C}_p(\mathbb{Q})$.

i) From lemma 2.11, we have $\exists\, N \in \mathbb{Z}_{\geq 0}$ such that $\forall\, n \geq N$, $|a_n|_p = |a_N|_p$. This proves that the given limit exits.

ii) By *i*) above, $\lim_{n\to\infty} |a_n|_p$ and $\lim_{n\to\infty} |b_n|_p$ both exist.

Thus, $0 = \lim_{n\to\infty} (|a_n|_p - |b_n|_p) = \lim_{n\to\infty} |a_n|_p - \lim_{n\to\infty} |b_n|_p$. The result follows.

□

Definition 2.13. *We define the field of p-adic numbers $\mathbb{Q}_p$, to be the following quotient,*

$$\mathbb{Q}_p := \mathscr{C}_p(\mathbb{Q})/\mathscr{N}_p(\mathbb{Q}).$$

Definition 2.14. *Let $|\cdot|_p \to \mathbb{R}_{\geq 0}$ be defined such that for $x \in \mathbb{Q}_p$, (x_n) a Cauchy sequence representing x, $|x|_p = \lim_{n\to\infty} |x_n|$.*

Note that this definition makes sense since in the lemmas above we have checked that the limit exists and that two representatives of the same point in $\mathbb{Q}_p$ have the same absolute value.

Proposition 2.15. *The function* $|\cdot|_p \to \mathbb{R}_{\geq 0}$ *is a non-archimedean absolute value on* $\mathbb{Q}_p$. *Moreover, the image of* $\mathbb{Q}_p$ *under* $|\cdot|_p$ *is equal to the image of* $\mathbb{Q}$ *under* $|\cdot|_p$.

Proof. Checking the axioms of a non-archimedean absolute value is left as an exercise.

Let $x \in \mathbb{Q}_p$, (x_n) a representative of x, then from lemma 2.11, $\exists\, N \in \mathbb{Z}_{\geq 0}$ such that $\forall\, n \geq N, |x_n|_p = |x_N|_p$. Thus $|x|_p = |x_N|_p$. The result follows. $\square$

We can consider an inclusion of $\omega : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$ by the mapping $q \mapsto (q) + \mathscr{N}_p(\mathbb{Q})$.

Proposition 2.16. *The image* $\omega(\mathbb{Q})$ *is dense in* $\mathbb{Q}_p$.

Proof. Let $x \in \mathbb{Q}_p$, $\epsilon > 0$. We will prove $\exists\, q \in \mathbb{Q}$ such that $|x - \omega(q)| < \epsilon$.

Let (x_n) be a representative of x and $0 < \epsilon' < \epsilon$.

Since (x_n) is a Cauchy sequence, $\exists\, N \in \mathbb{Z}_{\geq 0}$ such that for $\forall\, n, m \geq N$, $|x_n - x_m|_p < \epsilon'$.

Now, set $q = x_N$ and consider the element, $\omega(q)$ of $\mathbb{Q}_p$.

Then $\forall\, n \geq N$, we have $|x_n - q|_p = |x_n - x_N|_p < \epsilon'$.

Thus we have, $|x - \omega(q)|_p = \lim_{n\to\infty} |x_n - q|_p \leq \epsilon' < \epsilon$. $\square$

Proposition 2.17. $\mathbb{Q}_p$ *is complete with respect to* $|\cdot|_p$.

Proof. Let $(y_n)_{n=0}^{\infty}$ be a Cauchy sequence in $\mathbb{Q}_p$.

Define (q_n) in $\mathbb{Q}$ such that for each n, $|y_n - \omega(q_n)|_p < \frac{1}{n}$; this is possible for all n since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$.

Then since

$$|q_{n+1} - q_n|_p = |\omega(q_{n+1}) - \omega(q_n)|_p = |\omega(q_{n+1}) - y_{n+1} + y_{n+1} - y_n + y_n - \omega(q_n)|_p$$

we have

$$|q_{n+1} - q_n|_p \leq \texttt{max}\{|q_{n+1}) - y_{n+1}|_p, |y_{n+1} - y_n|_p, |y_n - \omega(q_n)|_p\} \to 0$$

as $n \to \infty$.

Thus (q_n) is a Cauchy sequence in $\mathbb{Q}$. Let $y = (q_n) + \mathscr{N}_p(\mathbb{Q})$.

Now claim: $\lim_{n\to\infty} y_n = y$.

Proof of claim:

Consider $|y_n - y|_p = |y_n - \omega(q_n) + \omega(q_n) - y|_p \leq \max\{|y_n - \omega(q_n)|_p, |\omega(q_n) - y|_p\} \to 0.$ □

From now on we will adopt the convention that we write $\omega(q)$ as just q for rational numbers q. By the context, one will be able to tell if we are considering q as an element of $\mathbb{Q}$ or $\mathbb{Q}_p$.

Proposition 2.18. *Let $x \in \mathbb{Q}_p$, $r \in \mathbb{Z}_{\geq 0}$. Then if $y \in x + p^r\mathbb{Z}_p$ we have*

$$y + p^r\mathbb{Z}_p = x + p^r\mathbb{Z}_p.$$

Proof. Let $z \in x + p^r\mathbb{Z}_p$ then,

$$|z - y|_p \geq \max\{|z - x|_p, |x - y|_p\} = p^r.$$

Thus, $z \in y + p^r\mathbb{Z}_p$ and so $x + p^r\mathbb{Z}_p \subseteq y + p^r\mathbb{Z}_p$. By symmetry we have the opposite inclusion and thus the result. □

Definition 2.19. *Let $\mathbb{K}$ be a field, we define $\nu : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ to be a valuation on $\mathbb{K}$ if the following hold.*

i) $\nu(ab) = \nu(a) + \nu(b)$ *for all* $a, b \in \mathbb{K}$*;*

ii) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ *for all* $a, b \in \mathbb{K}$*;*

iii) $\nu(a) = \infty \iff a = 0$.

Definition 2.20. *We define the p-adic valuation $\nu_p : \mathbb{Q}_p \to \mathbb{Q} \cup \{\infty\}$ such that for $x \in \mathbb{Q}_p$, $\nu_p(x)$ is the unique value such that $|x|_p = p^{-\nu_p(x)}$.*

It is worth noting for later work that by the definition of valuation, a sequence (a_n) in $\mathbb{Q}_p$ tends to 0 if and only if $\nu_p(a_n) \to \infty$.

Proposition 2.21. *ν_p is a valuation on $\mathbb{Q}_p$.*

Proof. i) Let $a, b \in \mathbb{Q}_p$ then since

$$p^{-\nu_p(ab)} = |ab|_p = |a|_p|b|_p = p^{-\nu_p(a)}p^{-\nu_p(b)} = p^{-(\nu_p(a)+\nu_p(b))}$$

we have $\nu_p(ab) = \nu_p(a) + \nu_p(b)$.

ii) Let $a, b \in \mathbb{Q}_p$ then since

$$p^{-\nu_p(a+b)} = |a + b|_p \leq \max\{|a|_p, |b|_p\} = \max\{p^{-\nu_p(a)}, p^{-\nu_p(b)}\} = p^{-\min\{\nu_p(a),\nu_p(b)\}}$$

we have $\nu_p(a + b) \geq \min\{\nu_p(a), \nu_p(b)\}$.

iii) $\nu_p(a) = \infty \iff |a|_p = 0 \iff a = 0$.

□

Definition 2.22. *We define the p-adic integers*

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid \nu_p(x) \geq 0\} = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}.$$

Proposition 2.23. *The image of $\mathbb{Z} \hookrightarrow \mathbb{Q}_p$, is dense in $\mathbb{Z}_p$.*

Proof. Let $x \in \mathbb{Z}_p$, $\epsilon > 0$. Then $\exists\, n \in \mathbb{Z}_{\geq 0}$ such that $p^{-n} \leq \epsilon$ and since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, $\exists\, \frac{a}{b} \in \mathbb{Q}$, where $\texttt{hcf}(a,b) = 1$, such that $|x - \frac{a}{b}|_p \leq p^{-n} < 1$.

Now since $|\frac{a}{b}|_p \leq \max\{|x|_p, |x - \frac{a}{b}|_p\} \leq 1$, we have that $\frac{a}{b} \in \mathbb{Z}_p$. Thus we have that $|\frac{a}{b}|_p \leq 1$, hence $p \nmid b$, and so $\exists\, c \in \mathbb{Z}$ such that $bc \equiv 1 \pmod{p^n}$.

Since $|\frac{a}{b} - ac|_p = |a|_p |\frac{1-bc}{b}|_p \leq p^{-n}$, and $ac \in \mathbb{Z}$, we have

$|x - ac|_p \leq \min\{|x - \frac{a}{b}|_p, |\frac{a}{b} - ac|_p\} \leq p^{-n} < \epsilon$. Allowing us to conclude. □

Corollary 2.24. *Let $n \in \mathbb{Z}_{\geq 0}$, then $n + \mathbb{Z}_{\geq 0}$ is dense in $\mathbb{Z}_p$.*

Proof. Follows using the same proof as Proposition 2.23, but we choose a to be non-negative and $c \in n + \mathbb{Z}_{\geq 0}$. □

Lemma 2.25. *Let $m \in \mathbb{Z}_{\geq 0}$, then $\mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}/p^m\mathbb{Z}$.*

Proof. Consider $\varphi : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}_p/p^m\mathbb{Z}_p$ such that $\varphi(a + p^m\mathbb{Z}) = a + p^m\mathbb{Z}_p$. Checking that φ is an isomorphism is left as an exercise. □

Proposition 2.26. *$\mathbb{Z}_p$ is compact.*

Proof. First we note that, since $\mathbb{Z}_p$ is closed in $\mathbb{Q}_p$ which is complete, $\mathbb{Z}_p$ is complete. Thus it is enough to show that $\mathbb{Z}_p$ is totally bounded.

Let $\epsilon > 0$, if $\epsilon \geq 1$ then $\{y : |y - a|_p < \epsilon\} \supset \mathbb{Z}_p$ for all $a \in \mathbb{Z}_p$. Now note that $|\cdot|_p$ only takes values in the set $\{p^z \mid z \in \mathbb{Z}\}$. Thus we need only consider $\epsilon = p^{-n}$, $n \geq 0$.

But since $\mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}/p^m\mathbb{Z}$ we have that

$$\mathbb{Z}_p = \bigcup_{i=1}^{p^n} (i + p^n\mathbb{Z}_p).$$

□

Proposition 2.27 (Legendre's Formula)**.** *For $n \in \mathbb{Z}_{>0}$, we have $\nu_p(n!) = \sum_{i=1}^{\infty} [\frac{n}{p^i}]$.*

Proof. $[\frac{n}{p}]$ counts the number of multiples of p less than n. But the multiples of p^2 have only been counted once so we need to add in $[\frac{n}{p^2}]$; but now the multiples of p^3 have only been counted twice so we must add $[\frac{n}{p^3}]$. Continuing this process we deduce the formula. We know that this sum is finite since $\exists\, s > 0$ such that $[\frac{n}{p^m}] = 0$, $\forall\, m > s$. □

2.2 Mahler's Theorem

Throughout this section we shall define L to be a finite extension of $\mathbb{Q}_p$. To define a valuation on L, we will use the following theorem.

Theorem 2.28. *Let $L : \mathbb{Q}_p$ be a finite extension of degree n, then the function $|\cdot| : L \to \mathbb{R}_{\geq 0}$ defined by $|x| = \sqrt[n]{|N_{L:\mathbb{Q}_p}(x)|_p}$ is a non-archimedean absolute value on L which extends the p-adic absolute value on $\mathbb{Q}_p$. Here $N_{L:\mathbb{Q}_p}(x)$ is the norm of the field extension $L : \mathbb{Q}_p$.*

Since this absolute value extends our absolute value on $\mathbb{Q}_p$ we shall denote it by $|x|_p$.

In addition L is complete with respect to $|x|_p$.

Proof. The proof of this result would take us too far from the aims of the project so we omit it. For the details see [Gou03]. □

We can then define a valuation on L as follows.

Definition 2.29. *Let $\nu_p : L \to \mathbb{R} \cup \{\infty\}$ be defined by $\nu_p(x) = -\frac{\log |x|_p}{\log p}$. Then this is a valuation on L.*

Checking that this is a valuation follows in the same way as checking ν_p is a valuation for $\mathbb{Q}_p$ (Proposition 2.21). We will now prove two propositions which show how the nice properties of $\mathbb{Q}_p$ can be carried over into finite field extensions of $\mathbb{Q}_p$.

Proposition 2.30. *Let $(a_k)_{k=1}^{\infty}$ be a sequence in L. Then the sequence is Cauchy if and only if $\nu_p(a_{n+1} - a_n) \to \infty$.*

Proof. The proof works in the say way as the proof of the equivalent theorem for non-archimedean absolute values (Proposition 2.7). □

Proposition 2.31. *Let $(a_k)_{k=1}^{\infty}$ be a sequence in L. Then the sum*

$$\sum_{k=1}^{\infty} a_k$$

converges if and only if $\lim_{k\to\infty} a_k = 0$.

Proof. First suppose that the sum converges, then in particular

$$\lim_{n\to\infty} \nu_p \left(\sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k \right) = \infty.$$

Hence, $\lim_{n\to\infty} \nu_p(a_n) = \infty$.

For the other direction we consider

$$S = \lim_{n,m\to\infty} \nu_p \left(\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right).$$

Without loss of generality we can consider $m < n$ then

$$S = \nu_p \left(\sum_{k=n+1}^{m} a_k \right) \geq \min_{n+1\leq k\leq m} \nu_p(a_k) \to \infty$$

as $n, m \to \infty$. $\square$

To begin our study of continuous functions on $\mathbb{Z}_p$, we need to develop some notation and concepts through the following definitions.

Definition 2.32. *Let B be an L-vector space, let $\{e_i \mid i \in I\}$ be a basis of B over L. We then define a valuation on B such that if $x \in B$, $x = \sum_{i\in I} a_i e_i$, $a_i \in L$ then $\nu_B(x) = \inf_{i\in I}\{\nu_p(a_i)\}$.*

Definition 2.33. *Let B be an L-vector space. From a valuation ν on B, we define a topology on B by defining the following sets to be open balls: $B(a, r) = \{x \in B \mid \nu(x - a) > r\}$ for any $a \in B$, $r \in \mathbb{R}$.*

Let B be an L-vector space, we say that B is L-Banach if B has a topology defined by a complete valuation on B.

Definition 2.34. *Let B_1, B_2 be L-Banach spaces. We say that a function $f : B_1 \to B_2$ is an L-Banach morphism if it is L-linear and continuous.*

It is an isometry of B_1 and B_2 if also f is bijective and $\nu_{B_2}(f(x)) = \nu_{B_1}(x)$, $\forall\, x \in B_1$.

Definition 2.35. *We define the set $l^0_\infty(I, L)$ to be the set of sequences $(a_n)_{n\in I}$ of elements of L that tend to 0.*

If $I = \mathbb{Z}_{\geq 0}$, $l^0_\infty(\mathbb{Z}_{\geq 0}, L)$ is the set of sequences $(a_n)_{n=0}^{\infty}$ in L such that $\lim_{n\to\infty} a_n = 0$.

Definition 2.36. *Let B be an L-Banach space. We say a collection of elements $(e_i)_{i\in I}$ of B is an orthonormal basis of B if the function $\phi : l^0_\infty(I, L) \to B$ defined by $\phi((a_i)_{i\in I}) = \sum_{i\in I} a_i e_i$, is an isometry.*

Definition 2.37. *We define $\mathscr{C}^0(\mathbb{Z}_p, L)$ to be the set of all continuous functions $\phi : \mathbb{Z}_p \to L$.*

We define the valuation $\nu_{\mathscr{C}^0} : \mathscr{C}^0(\mathbb{Z}_p, L) \to \mathbb{R} \cup \{\infty\}$ such that for $\phi \in \mathscr{C}^0(\mathbb{Z}_p, L)$, $\nu_{\mathscr{C}^0}(\phi) = \inf_{x\in\mathbb{Z}_p} \nu_p(\phi(x))$.

Proposition 2.38. *The valuation $\nu_{\mathscr{C}^0}$ as defined above is in fact a valuation on $\mathscr{C}^0$.*

Proof. We check each property separately.

i) Let $\phi, \psi \in \mathscr{C}^0$, then

$$\begin{aligned}\nu_{\mathscr{C}^0}(\phi\psi) = \inf_{x\in\mathbb{Z}_p} \nu_p(\phi\psi(x)) &= \inf_{x\in\mathbb{Z}_p} \nu_p(\phi(x))\nu_p(\psi(x)) \\ &= \left(\inf_{x\in\mathbb{Z}_p} \nu_p(\phi(x))\right)\left(\inf_{x\in\mathbb{Z}_p} \nu_p(\psi(x))\right) \\ &= \nu_{\mathscr{C}^0}(\phi)\nu_{\mathscr{C}^0}(\psi).\end{aligned}$$

ii) Let $\phi, \psi \in \mathscr{C}^0$, then

$$\begin{aligned}\nu_{\mathscr{C}^0}(\phi+\psi) = \inf_{x\in\mathbb{Z}_p} \nu_p(\phi+\psi(x)) &\leq \inf_{x\in\mathbb{Z}_p} (\nu_p(\phi(x)) + \nu_p(\psi(x))) \\ &= \inf_{x\in\mathbb{Z}_p} \nu_p(\phi(x)) + \inf_{x\in\mathbb{Z}_p} \nu_p(\psi(x)) \\ &= \nu_{\mathscr{C}^0}(\phi) + \nu_{\mathscr{C}^0}(\psi).\end{aligned}$$

iii) Let ϕ be the zero map, then

$$\nu_{\mathscr{C}^0}(\phi) = \inf_{x\in\mathbb{Z}_p} \nu_p(\phi(x)) = \inf_{x\in\mathbb{Z}_p} \nu_p(0) = \infty.$$

□

Definition 2.39. *For $n \in \mathbb{Z}_{\geq 0}$, we define the binomial polynomial $\binom{x}{n} : \mathbb{Q}_p \to \mathbb{Q}_p$ to be the polynomial,* $\binom{x}{n} = \begin{cases} 1 & \text{if } n = 0 \\ \frac{x(x-1)\dots(x-n+1)}{n!} & \text{if } n \geq 1. \end{cases}$

Note that since $\binom{x}{n}$ is a polynomial, it is continuous, i.e $\binom{x}{n} \in \mathscr{C}^0(\mathbb{Z}_p, L)$.

We will soon realise that the binomial polynomial plays a crucial role in the work of continuous functions and we begin this with the results below.

Lemma 2.40. *For all $x \in \mathbb{Z}_p$, $n \in \mathbb{Z}_{\geq 1}$, we have $\binom{x}{n} = \binom{x-1}{n} + \binom{x-1}{n-1}$.*

Proof. The result is trivial for $x \in \mathbb{Z}$ and since both sides of the equation above are continuous on $\mathbb{Z}_p$ and equal on the dense subset $\mathbb{Z} \subset \mathbb{Z}_p$, we have equality. □

Proposition 2.41. $\nu_{\mathscr{C}^0}(\binom{x}{n}) = 0$, $\forall\, n \in \mathbb{Z}_{\geq 0}$.

Proof. We first note that $\nu_{\mathscr{C}^0}$ is defined by, for $\phi \in \mathscr{C}^0(\mathbb{Z}_p, L)$, $\nu_{\mathscr{C}^0}(\phi) = \inf_{x\in\mathbb{Z}_p} \phi(x)$.

Since $\binom{n}{n} = 1$ we have $\nu_{\mathscr{C}^0}(\binom{x}{n}) \leq 0$.

Now consider $\binom{n+k}{n}$ for $k \in \mathbb{Z}_{\geq 0}$. $\binom{n+k}{n}$ is the number of ways to choose n items from a collection of $n+k$ objects, thus $\binom{n+k}{n} \in \mathbb{Z}$ and so $\nu_{\mathscr{C}^0}(\binom{n+k}{n}) \geq 0$, $\forall\, k \in \mathbb{Z}_{\geq 0}$.

Now since $n + \mathbb{Z}_{\geq 0}$ is dense in $\mathbb{Z}_p$, we have that $\nu_{\mathscr{C}^0}(\binom{x}{n}) \geq 0$, for all $x \in \mathbb{Z}_p$. □

We now consider the main theorem of this section, Mahler's Theorem, which gives a characterisation of all continuous functions on $\mathbb{Z}_p$ in terms of a series based on binomial polynomials. The coefficients of this series can be formally calculated using the discrete derivative which we define next.

Definition 2.42. *Let $\phi : \mathbb{Q}_p \to L$, then define the k^{th} discrete derivative $\phi^{[k]}$ of ϕ by induction:*

i) $\phi^{[0]} = \phi$;

ii) for $k \geq 1$ and $x \in \mathbb{Q}_p$, we have $\phi^{[k+1]}(x) = \phi^{[k]}(x+1) - \phi^{[k]}(x)$.

If $n \in \mathbb{Z}_{>0}$, we define the n^{th} Mahler coefficient $a_n(\phi) := \phi^{[n]}(0)$.

Theorem 2.43 (Mahler's Theorem)**.** *Let $\phi \in \mathscr{C}^0(\mathbb{Z}_p, L)$.*

i) If $\phi \in \mathscr{C}^0(\mathbb{Z}_p, L)$, then

a) $\lim_{n\to\infty} a_n(\phi) = 0$;

b) $\sum_{n=0}^{\infty} a_n(\phi)\binom{x}{n} = \phi(x)\ \forall\, x \in \mathbb{Z}_p$.

ii) The map $\phi \mapsto (a_n(\phi))_{n\in N}$ is an isometry of $\mathscr{C}^0(\mathbb{Z}_p, L)$ on $l^0_\infty(\mathbb{Z}_{\geq 0}, L)$.

Given a continuous function we say that the sum as given in b) is its Mahler expansion.

Proof. While the proof of Mahler's theorem is not hugely complicated it is long and, for the most part, not illuminating. I have therefore omitted it, but give two references for different proofs, namely [Col10] and [Con]. ☐

Lemma 2.44. *Suppose $\phi(x) = \sum_{n=0}^{\infty} c_n \binom{x}{n}$ where $(c_n)_{n=0}^{\infty} \in l^0_\infty(\mathbb{Z}_{\geq 0}, L)$. Then*

$$a_n(\phi) = c_n.$$

Proof. Consider

$$\begin{aligned}\phi^{[1]}(x) &= \sum_{n=0}^{\infty} c_n \left(\binom{x+1}{n} - \binom{x}{n}\right)\\ &= \sum_{n=1}^{\infty} c_n \binom{x}{n-1}\\ &= \sum_{n=0}^{\infty} c_{n+1} \binom{x}{n}\end{aligned}$$

since $\binom{x+1}{0} - \binom{x}{0} = 0$. Thus by induction we deduce that for all $k \in \mathbb{Z}_{\geq 0}$,

$$\phi^{[k]}(x) = \sum_{n=0}^{\infty} c_{n+k} \binom{x}{n}.$$

Thus we can conclude $a_n(\phi) = \phi^{[n]}(0) = c_n$. ☐

Corollary 2.45. *The set* $\{\binom{x}{n} \mid n \in \mathbb{Z}_{\geq 0}\}$ *is an orthonormal basis of* $\mathscr{C}^0(\mathbb{Z}_p, L)$.

Proof. Clear from Mahler's Theorem. □

2.3 Locally Analytic Functions

Definition 2.46. *Let* $a \in \mathbb{Q}_p$, $h \in \mathbb{Z}_{\geq 0}$. *We say that a function* $\phi : a + p^h\mathbb{Z}_p \to L$ *is* L*-analytic on* $a + p^h\mathbb{Z}_p$ *if there exists a sequence* $a_k(\phi, a)$, $k \in \mathbb{Z}_{\geq 0}$, *of elements of* L *such that* $\nu_p(a_k(\phi, a)) + kh \to \infty$, *and* $\phi(x) = \sum_{k=0}^{\infty} a_k(\phi, a)(x - a)^k$, $\forall\, x \in a + p^h\mathbb{Z}_p$.

We define the valuation $\nu_{a+p^h\mathbb{Z}_p}$ *for analytic functions on* $a + p^h\mathbb{Z}_p$ *to be*

$$\nu_{a+p^h\mathbb{Z}_p}(\phi) = \inf_{k\in\mathbb{Z}_{\geq 0}} \nu_p(a_k(\phi, a)) + kh$$

for ϕ *an analytic function on* $a + p^h\mathbb{Z}_p$.

Definition 2.47. *Let* $h \in \mathbb{Z}_{\geq 0}$, *we say* $\phi : \mathbb{Z}_p \to L$ *is* L*-analytic of order* h *if* $\phi \mid_{a+p^h\mathbb{Z}_p}$ *is an* L*-analytic function on* $a + p^h\mathbb{Z}_p, \forall\, a \in \mathbb{Z}_p$.

We will let $LA_h(\mathbb{Z}_p, L)$ *denote the set of locally analytic functions of order* h *and endow* $LA_h(\mathbb{Z}_p, L)$ *with the valuation* ν_{LA_h}, *defined by*

$$\nu_{LA_h}(\phi) = \inf_{a\in\mathbb{Z}_p} \nu_{a+p^h\mathbb{Z}_p}(\phi_{a,h})$$

where $\phi \in LA_h(\mathbb{Z}_p, L)$ *and* $\phi_{a,h} = \phi \mid_{a+p^h\mathbb{Z}_p}$.

We define $LA(\mathbb{Z}_p, L) = \bigcup_{h\in\mathbb{Z}_{\geq 0}} LA_h(\mathbb{Z}_p, L)$.

Theorem 2.48 (Amice's Theorem)**.** *The set* $\{[\frac{n}{p^h}]!\binom{x}{n} \mid n \in \mathbb{Z}_{\geq 0}\}$ *is an orthonormal basis of* $LA_h(\mathbb{Z}_p, L)$.

Proof. See [Col10]. □

The corollary below further shows the power of using Mahler expansions, stating that we can say if a function is locally analytic or not by considering its Mahler expansion.

Corollary 2.49. *If* $\phi \in \mathscr{C}^0(\mathbb{Z}_p, L)$, *then the following are equivalent:*

i) $\phi \in LA(\mathbb{Z}_p, L)$;

ii) $\liminf \frac{1}{n}\nu_p(a_n(\phi)) > 0$.

Proof. The proof of this result is sketched in [Col10] but I have adapted the proof to make it easier to understand. First assume that $\phi \in LA(\mathbb{Z}_p, L)$, then $\exists\, h \in \mathbb{Z}_{\geq 0}$ such that $\phi \in LA_h(\mathbb{Z}_p, L)$. From Amice's Theorem we have that $\exists\, (c_n)_{n=0}^{\infty} \in l_{\infty}^0(\mathbb{Z}_{\geq 0}, L)$ such that $\phi(x) = \sum_{n=1}^{\infty} c_n[\frac{n}{p^h}]!\binom{x}{n}$.

Then from Lemma 2.31 we see that $a_n(\phi) = c_n[\frac{n}{p^h}]!$ for all $n \in \mathbb{Z}_{\geq 0}$.

Therefore

$$\liminf_{n\to\infty} \frac{1}{n}\nu_p(a_n(\phi)) = \liminf_{n\to\infty} \frac{1}{n}\left(\nu_p(c_n) + \nu_p\left(\left[\frac{n}{p^h}\right]!\right)\right).$$

We now apply Legendre's formula and the fact that $(c_n) \to 0$, i.e that $\nu_p(c_n) \to \infty$.

Thus we deduce that

$$\liminf_{n\to\infty} \frac{1}{n}\nu_p(a_n(\phi)) \geq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{\infty}\left[\frac{\left[\frac{n}{p^h}\right]}{p^i}\right]$$

$$\geq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{\infty}\frac{n}{p^{h+i+2}} = \sum_{i=0}^{\infty}\frac{1}{p^{h+i+3}} = \frac{1}{p^{h+2}(p-1)} > 0.$$

For the other direction suppose that $\liminf \frac{1}{n}\nu_p(a_n(\phi)) > 0$, then $\exists\, h \in \mathbb{Z}_{\geq 0}$ such that

$$\liminf \frac{1}{n}\nu_p(a_n(\phi)) > \frac{1}{(p-1)p^{h-1}}.$$

Then define $c_n \in L$ such that $c_n = \left(\left[\frac{n}{p^h}\right]!\right)^{-1} a_n(\phi)$ for all $n \in \mathbb{Z}_{\geq 0}$.

We calculate

$$\liminf_{n\to\infty} \frac{1}{n}\nu_p(c_n) \geq \liminf_{n\to\infty}\left(\frac{1}{(p-1)p^h} - \frac{1}{n}\nu_p\left(\left[\frac{n}{p^h}\right]!\right)\right).$$

Now using Legendre's formula we have that

$$\nu_p\left(\left[\frac{n}{p^h}\right]!\right) \leq \sum_{i=1}^{\infty}\frac{n}{p^{h+i}} = \frac{n}{(p-1)p^h}$$

hence,

$$\liminf \frac{1}{n}\nu_p(c_n) \geq \frac{1}{(p-1)p^{h-1}} - \frac{n}{(p-1)p^h} > 0$$

and from this we deduce that $\nu_p(c_n) \to \infty$.

So $(c_n)_{n=1}^{\infty} \in l_\infty^0(\mathbb{Z}_{\geq 0}, L)$ such that $\phi(x) = \sum_{n=1}^{\infty} c_n \left[\frac{n}{p^h}\right]!\binom{x}{n}$. □

2.4 $\mathscr{C}^r$ Functions

Definition 2.50. *Let $\phi : \mathbb{Z}_p \to L$, we say that ϕ is differentiable at $x_0 \in \mathbb{Z}_p$ if the limit $\lim\limits_{n\to\infty} \frac{\phi(x_0+h)-\phi(x_0)}{h}$ exists.*

This limit is then denoted $\phi'(x_0)$; also let $\phi^{(j)}$ denote the $j^{\mathtt{th}}$ derivative of ϕ.

We say that ϕ is differentiable of order 1 if it is differentiable at every $x_0 \in \mathbb{Z}_p$.

We say that ϕ is differentiable of order r if its $(r-1)^{\mathtt{st}}$ derivative is differentiable of order 1.

Definition 2.51. *Let $\phi : \mathbb{Z}_p \to L$, then if $\phi^{(j)}$ exists for $0 \leq k \leq [r]$, we define $\epsilon_{\phi,r} : \mathbb{Z}_p \times \mathbb{Z}_p \to L$ and $C_{\phi,r} : \mathbb{Z}_{\geq 0} \to \mathbb{R} \cup \{\infty\}$ by*

$$\epsilon_{\phi,r}(x,y) = \phi(x+y) - \sum_{j=0}^{[r]} \phi^{(j)}(x) \frac{y^j}{j!}$$

and

$$C_{\phi,r}(h) = \inf_{x \in \mathbb{Z}_p, y \in p^h \mathbb{Z}_p} \nu_p(\epsilon_{\phi,r}(x,y)) - rh.$$

Definition 2.52. *If $r \geq 0$, we define $\phi : \mathbb{Z}_p \to L$ to be a $\mathscr{C}^r$ function if $\phi^{(j)}$ exists for $0 \leq j \leq [r]$ and $C_{\phi,r}(h) \to \infty$ as $h \to \infty$.*

Definition 2.53. *We define $\mathscr{C}^r(\mathbb{Z}_p, L)$ to be the set of functions $\phi : \mathbb{Z}_p \to L$ which are $\mathscr{C}^r$. We endow $\mathscr{C}^r(\mathbb{Z}_p, L)$ with the valuation defined, for $\phi \in \mathscr{C}^r(\mathbb{Z}_p, L)$, by*

$$\nu'_{\mathscr{C}^r}(\phi) = \min\left\{ \inf_{0 \leq j \leq [r], x \in \mathbb{Z}_p} \nu_p\left(\frac{\phi^{(j)}(x)}{j!}\right), \inf_{x,y \in \mathbb{Z}_p} \nu_p(\epsilon_{\phi,r}(x,y)) - r\nu_p(y) \right\}.$$

Defining this valuation makes $\mathscr{C}^r(\mathbb{Z}_p, L)$ into an L-Banach space.

Proposition 2.54. *If $r \geq 1$ and if $\phi \in \mathscr{C}^r(\mathbb{Z}_p, L)$, then ϕ is differentiable at all $x \in \mathbb{Z}_p$, and furthermore,*

i) $\phi' \in \mathscr{C}^{r-1}(\mathbb{Z}_p, L)$ and $\exists\, C_0(r) \in \mathbb{R}$ such that

$$\nu'_{\mathscr{C}^{r-1}}(\phi') \geq \nu'_{\mathscr{C}^{r-1}}(\phi) - C_0(r)$$

for all $\phi \in \mathscr{C}^r(\mathbb{Z}_p, L)$;

ii) $(\phi')^{(j)} = \phi^{(j+1)}$ if $j \leq r$.

Proof. See [Col10]. □

Corollary 2.55. *If $r \geq 1$ and if $\phi \in \mathscr{C}^r(\mathbb{Z}_p, L)$, then for $j \leq [r]$*

$$C_{\phi^{(j)}, r-j}(h) \geq C_{\phi,r} - C([r])$$

where $C(N) := \sum_{n=1}^{N} \nu_p(n!)$.

Proof. See [Col10]. □

2.5 Comparing $\mathscr{C}^r(Z_p, L)$ and $LA(\mathbb{Z}_p, L)$

Proposition 2.56. *If $h \in \mathbb{Z}_{\geq 0}$ and if $r > 0$, then $LA_h(\mathbb{Z}_p, L) \subset \mathscr{C}^r(\mathbb{Z}_p, L)$.*

Moreover, if $\phi \in LA_h(\mathbb{Z}_p, L)$, then $\nu'_{\mathscr{C}^r}(\phi) \geq \nu_{LA_h}(\phi) - rh$.

Proof. Colmez gives a sketch of this proof in [Col10] and I have filled in the details. Let $\phi \in LA_h(\mathbb{Z}_p, L)$ and consider $\nu_{LA_h}(\phi) - hj$ for $x \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_{\geq 0}$.

Then $\nu_{LA_h}(\phi) - hj = \inf_{a\in\mathbb{Z}_p}(\inf_{k\in\mathbb{Z}_{\geq 0}} \nu_p(a_k(\phi, a)) + kh) - hj \leq \inf_{a\in\mathbb{Z}_p}(\nu_p(a_j(\phi, a))$.

Now, since

$$\frac{\phi^{(j)}(x)}{j!}\Big|_{a+p^h\mathbb{Z}_p} = a_j(\phi, a) + \sum_{k\geq j+1}^{\infty} \binom{k}{j} a_k(\phi, a)(x-a)^{k-j},$$

we then have

$$\nu_p(a_j(\phi, a)) \leq \nu_p\left(\frac{\phi^{(j)}(x)}{j!}\right).$$

Hence, $\nu_{LA_h}(\phi) - hj \leq \nu_p\left(\frac{\phi^{(j)}(x)}{j!}\right)$.

Now consider $\phi(x+y)$ where $\nu_p(y) \geq h$. Since ϕ is a locally analytic function of order h and $y \in B(x, h)$, then $\exists\, a_k(\phi, x), k \in \mathbb{Z}_{\geq 0}$ such that $\nu_p(a_k(\phi, x)) + kh \to \infty$ and $\phi(z) = \sum_{k=0}^{\infty} a_k(\phi, x)(z-x)^k$, $\forall\, z \in B(x, h)$.

By differentiating k times at the point $x+y$ we deduce that $a_k(\phi, x) = \frac{\phi^{(j)}(x)}{j!}$.

Thus $\phi(x+y) = \sum_{j=0}^{\infty} \phi^{(j)}(x)\frac{y^j}{j!}$.

This allows us to write, $\epsilon_{\phi,r}(x, y) = \begin{cases} \sum_{j>r} \frac{\phi^{(j)}(x)}{j!} y^j & \text{if } \nu_p(y) \geq h \\ \phi(x+y) - \sum_{j=0}^{[r]} \frac{\phi^{(j)}(x)}{j!} y^j & \text{if } \nu_p(y) < h \end{cases}$

Thus if $\nu_p(y) \geq h$, we have

$$\nu_p(\epsilon_{\phi,r}(x, y)) = \min_{j>r}\left\{\nu_p\left(\frac{\phi^{(j)}(x)}{j!}\right) + j\nu_p(y)\right\} \geq \min_{j>r}\left\{\nu_{LA_h}(\phi) - hj + j\nu_p(y)\right\}.$$

We then deduce the lower bound $\nu_p(\epsilon_{\phi,r}(x, y)) \geq \nu_{LA_h}(\phi) + ([r]+1)(\nu_p(y) - h)$.

Thus

$$C_{\phi,r}(h) \geq \nu_{LA_h}(\phi) + ([r]+1)(\nu_p(y) - h) - rh \geq \nu_{LA_h}(\phi) + ([r]+1)(\nu_p(y) - h) - r\nu_p(y)$$

for all $y \in p^h\mathbb{Z}_p$. Letting $\nu_p(y) \to \infty$, we see that $\phi \in \mathscr{C}^r(\mathbb{Z}_p, L)$.

To deduce the inequality, we first consider

$$\nu_p(\phi^{(j)}(x)) = \min\left\{\inf_{0\leq j\leq [r], x\in\mathbb{Z}_p} \nu_p\left(\frac{\phi^{(j)}(x)}{j!}\right), \inf_{x,y\in\mathbb{Z}_p} \nu_p(\epsilon_{\phi,r}(x, y)) - r\nu_p(y)\right\}.$$

Then, considering the first infimum in the minimum above,

$$\inf_{0\leq j\leq [r], x\in\mathbb{Z}_p} \nu_p\left(\frac{\phi^{(j)}(x)}{j!}\right) \geq \inf_{0\leq j\leq [r], x\in\mathbb{Z}_p} \nu_{LA_h}(\phi) - hj \geq \nu_{LA_h}(\phi) - rh.$$

We deal with the second infimum in two cases. Firstly, if $\nu_p(y) \geq h$, then

$$\begin{aligned}\inf_{x,y\in\mathbb{Z}_p} \nu_p(\epsilon_{\phi,r}(x,y)) - r\nu_p(y) &\geq \inf_{x,y\in\mathbb{Z}_p}\left(\inf_{j>r} \nu_p\left(\frac{\phi^{(j)}(x)}{j!}y^j\right) - r\nu_p(y)\right) \\ &\geq \inf_{x,y\in\mathbb{Z}_p}\left(\inf_{j>r}(\nu_{LA_h}(\phi) - hj + j\nu_p(y)) - r\nu_p(y)\right).\end{aligned}$$

Since $\nu_p(y) \geq h$, we have

$$\inf_{x,y\in\mathbb{Z}_p} \nu_p(\epsilon_{\phi,r}(x,y)) - r\nu_p(y) \geq \inf_{x,y\in\mathbb{Z}_p} (\nu_{LA_h}(\phi) - hr + r\nu_p(y)) - r\nu_p(y)) = \nu_{LA_h}(\phi) - hr$$

Lastly we consider the case $\nu_p(y) < h$, then

$$\inf_{x,y\in\mathbb{Z}_p} \nu_p(\epsilon_{\phi,r}(x,y)) - r\nu_p(y) \geq \inf_{x,y\in\mathbb{Z}_p}\left(\min\{\nu_{LA_h}(\phi), \inf_{0\leq j\leq [r]}(\nu_{LA_h}(\phi) - hj + j\nu_p(y)) - r\nu_p(y)\}\right)$$

Since $\nu_p(y) < h$, we have

$$\begin{aligned}\inf_{x,y\in\mathbb{Z}_p} \nu_p(\epsilon_{\phi,r}(x,y)) - r\nu_p(y) &\geq \inf_{x,y\in\mathbb{Z}_p} (\min\{\nu_{LA_h}(\phi), \nu_{LA_h}(\phi) - hr + r\nu_p(y)\} - r\nu_p(y)) \\ &= \inf_{x,y\in\mathbb{Z}_p} (\nu_{LA_h}(\phi) - hr + r\nu_p(y) - r\nu_p(y)) \\ &= \nu_{LA_h}(\phi) - hr.\end{aligned}$$

Thus we deduce the inequality $\nu'_{\mathscr{C}^r}(\phi) \geq \nu_{LA_h}(\phi) - rh$. □

It is possible to prove a result similar to Corollary 2.49 concerning the Mahler coefficients of $\mathscr{C}^r$, for which we first need the following definition.

Definition 2.57. *If $i \in \mathbb{Z}_{\geq 0}$, we denote $l(i)$ to be the smallest integer n such that $p^n > i$.*

Thus $l(0) = 0$ and $l(i) = [\frac{\log i}{\log p}] + 1$ if $i \geq 1$.

Theorem 2.58. *If $r > 0$ and $\phi : \mathbb{Z}_p \to L$, we can write $\phi = \sum_{n=0}^{\infty} a_n(\phi)\binom{x}{n}$, the Mahler decomposition of ϕ.*

Then $\phi \in \mathscr{C}^r(\mathbb{Z}_p, L)$ if and only if $v_p(a_n(\phi)) - rl(n) \to \infty$ as $n \to \infty$.

Proof. See [Col10]. □

Definition 2.59. *We define $\mathscr{C}^\infty(\mathbb{Z}_p, L) := \bigcap_{r>0} \mathscr{C}^r(\mathbb{Z}_p, L)$.*

Proposition 2.60. $\mathscr{C}^\infty(\mathbb{Z}_p, L) \neq LA(\mathbb{Z}_p, L)$.

Proof. We consider the function $\phi(x) = \sum_{n=0}^{\infty} p^{[\sqrt{nl(n)}]}\binom{x}{n}$; since this function is written in the form of a Mahler expansion, we can determine its properties by considering its coefficients $a_n(\phi) = p^{[\sqrt{nl(n)}]}$.

First we consider

$$\liminf \frac{1}{n}\nu_p(a_n(\phi)) = \liminf \frac{1}{n}[\sqrt{nl(n)}] \leq \liminf \frac{1}{n}\sqrt{nl(n)} = 0$$

thus $\phi \notin LA(\mathbb{Z}_p, L)$ by Corollary 2.49.

Now consider

$$v_p(a_n(\phi)) - rl(n) = [\sqrt{nl(n)}] - rl(n) \geq \sqrt{nl(n)} - rl(n) = \sqrt{l(n)}(\sqrt{n} - r\sqrt{l(n)}) \to \infty$$

hence $\phi \in \mathscr{C}^\infty(\mathbb{Z}_p, L)$ by Theorem 2.58. □

2.6 Locally Polynomial Functions

Definition 2.61. *Let I be a subset of $\mathbb{Z}_{\geq 0}$ then if $h \in \mathbb{Z}_{\geq 0}$, we denote $LP_h^I(\mathbb{Z}_p, L)$ the set of functions $\phi : \mathbb{Z}_p \to L$ such that for any $a \in \mathbb{Z}_p$ $\exists\, a_i \in \mathbb{Z}_p$ with $\phi\mid_{a+p^h\mathbb{Z}_p} = \sum_{i \in I} a_i x^i$.*

We define $LP^I(\mathbb{Z}_p, L)$ to be the set of functions $\phi : \mathbb{Z}_p \to L$ such that, for all $a \in \mathbb{Z}_p$, there exists $h \in \mathbb{Z}_{\geq 0}$ such that $\phi\mid_{a+p^h\mathbb{Z}_p} = \sum_{i \in I} a_i x^i$ for some $a_i \in \mathbb{Z}_p$.

If $I \subset \mathbb{R}$, we denote $LP_h^I(\mathbb{Z}_p, L)$ and $LP^I(\mathbb{Z}_p, L)$ to be the spaces $LP_h^{I\cap\mathbb{Z}_{\geq 0}}(\mathbb{Z}_p, L)$ and $LP^{I\cap\mathbb{Z}_{\geq 0}}(\mathbb{Z}_p, L)$ respectively.

Proposition 2.62. $LP^I(\mathbb{Z}_p, L) = \bigcup_{h \in \mathbb{Z}_{\geq 0}} LP_h^I(\mathbb{Z}_p, L)$.

Proof. It is clear that $LP^I(\mathbb{Z}_p, L) \supset \bigcup_{h \in \mathbb{Z}_{\geq 0}} LP_h^I(\mathbb{Z}_p, L)$.

Now let $\phi \in LP^I(\mathbb{Z}_p, L)$, then $\forall\, a \in \mathbb{Z}_p$ there exists $h_a \in \mathbb{Z}_{\geq 0}$ such that

$$\phi\mid_{a+p^{h_a}\mathbb{Z}_p} = \sum_{i \in I} a_i x^i.$$

Thus $\{a + p^{h_a} \mid a \in \mathbb{Z}_p\}$ is an open cover of $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is compact, there exists a finite set J such that $\{a_j + p^{h_{a_j}} \mid j \in J\}$ is a finite cover of $\mathbb{Z}_p$. Let $h = \min_{j \in J}\{h_{a_j}\}$.

Then for any $\alpha \in \mathbb{Z}_p$, $\alpha \in a_j + p^{h_{a_j}}$ for some $j \in J$ such that $\phi\mid_{a+p^{h_{a_j}}\mathbb{Z}_p} = \sum_{i \in I} a_i x^i$. Thus

$$\phi\mid_{a+p^h\mathbb{Z}_p} = \sum_{i \in I} a_i x^i.$$

So $\phi \in LP_h^I(\mathbb{Z}_p, L)$. □

Definition 2.63. *If $i \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$, we define $e_{i,k,r} \in LP^{[0,r]}$ to be the function*

$$e_{i,k,r} := p^{[l(i)r]} \mathbb{1}_{i+p^{l(i)}\mathbb{Z}_p}(x) \left(\frac{x-i}{p^{l(i)}} \right)^k .$$

Proposition 2.64. *i) The set $\{e_{i,k,r} \mid 0 \leq i \leq p^h - 1, 0 \leq k \leq r\}$ is a basis for $LP_h^{[0,r]}(\mathbb{Z}_p, L)$.*

ii) The set $\{e_{i,k,r} \mid i \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq r\}$ is a basis for $LP^{[0,r]}(\mathbb{Z}_p, L)$.

Proof. See [Col10]. □

Proposition 2.65. *Let $r \geq 0$. If $\phi \in \mathscr{C}^r(\mathbb{Z}_p, L)$ and $h \in \mathbb{Z}_{\geq 0}$, let $\phi_n \in LP_h^{[0,r]}$ be the function*

$$\phi_h(x) = \sum_{i=0}^{p^h-1} \mathbb{1}_{i+p^h\mathbb{Z}_p}(x) \left(\sum_{k=0}^{[r]} \frac{\phi^{(k)}(i)}{k!}(x-i)^k \right) .$$

Then

i) $\nu_{LA_{h+1}}(\phi_{h+1} - \phi_h) \geq rh + C_{\phi,r}(h) - C([r])$;

ii) ϕ_h tends to ϕ on $\mathscr{C}^r(\mathbb{Z}_p, L)$.

Proof. Colmez gives a sketch of this proof in [Col10] but I have filled in the details. We will begin by assuming *i*) holds, then since $C_{\phi,r(h)} \to \infty$ as $h \to \infty$, and using Proposition 2.56, we deduce that (ϕ_h) has a limit in $\mathscr{C}^r(\mathbb{Z}_p, L)$.

Moreover, since $\phi_h(i) = \phi(i)$ for all $i \in \mathbb{Z}_{\geq 0}$ and $i \leq p^h - 1$, the limit of ϕ_h coincides with ϕ on $\omega(\mathbb{Z}_{\geq 0}) \subset \mathbb{Z}_p$. Thus the two are equal since $\mathbb{Z}_{\geq 0}$ is dense in $\mathbb{Z}_p$. So *ii*) holds. It remains to prove *i*).

First note the following fact: if $i \in \mathbb{Z}_p$ and $h \in \mathbb{Z}_{\geq 0}$, then

$$i + p^h\mathbb{Z}_p = i + p^h \left(\bigcup_{a=0}^{p-1} (i + p\mathbb{Z}_p) \right) = \bigcup_{a=0}^{p-1} i + ap^h + p^{h+1}\mathbb{Z}_p.$$

Thus we have

$$\phi_{h+1}(x) - \phi_h(x) = \sum_{i=0}^{p^h-1} \sum_{a=0}^{p-1} \mathbb{1}_{i+ap^h+p^{h+1}\mathbb{Z}_p} \sum_{k=0}^{[r]} \left(\phi^{(k)}(i+ap^h) \frac{(x-i-ap^h)^k}{k!} - \phi^{(k)}(i) \frac{(x-i)^k}{k!} \right) .$$

Now consider

$$\sum_{k=0}^{[r]} \phi^{(k)}(i)\frac{(x-i)^k}{k!} = \sum_{k=0}^{[r]} \phi^{(k)}(i)\frac{(x-i-ap^h+ap^h)^k}{k!}$$
$$= \sum_{k=0}^{[r]}\sum_{m=0}^{k} \phi^{(k)}(i)\binom{k}{m}\frac{(x-i-ap^h)^m(ap^h)^{k-m}}{k!}.$$

Changing the order of summation, we deduce

$$\sum_{k=0}^{[r]} \phi^{(k)}(i)\frac{(x-i)^k}{k!} = \sum_{j=0}^{[r]}\sum_{l=0}^{[r]-j} \phi^{(j+l)}(i)\binom{j+l}{j}\frac{(x-i-ap^h)^j(ap^h)^{j+l-j}}{(j+l)!}$$
$$= \sum_{j=0}^{[r]}\sum_{l=0}^{[r]-j} \phi^{(j+l)}(i)\frac{(x-i-ap^h)^j(ap^h)^l}{j!l!}.$$

Thus we have

$$\sum_{k=0}^{[r]}\left(\phi^{(k)}(i+ap^h)\frac{(x-i-ap^h)^k}{k!} - \phi^{(k)}(i)\frac{(x-i)^k}{k!}\right)$$
$$= \sum_{j=0}^{[r]}\frac{p^{j(h+1)}}{j!}\left(\phi^{(j)}(i+ap^h) - \sum_{l=0}^{[r]-j}\phi^{(j+l)}(i)\frac{(ap^h)^l}{l!}\right)\left(\frac{x-i-ap^h}{p^{h+1}}\right)^j.$$

Now, using Corollary 2.55, we deduce

$$\nu_p\left(\phi^{(j)}(i+ap^h) - \sum_{l=0}^{[r]-j}\phi^{(j+l)}(i)\frac{(ap^h)^l}{l!}\right) \geq C_{\phi^{(j)},r-j}+(r-j)h \geq C_{\phi,r}(h)-C([r])+(r-j)h.$$

Hence

$$\nu_{LA_{h+1}}(\phi_{h+1}-\phi_h) \geq \inf_{j\leq[r]}\left(\nu_p\left(\frac{p^j(h+1)}{j!}\right) + C_{\phi,r}(h) - C([r]) + (r-j)h\right)$$
$$= rh + C_{\phi,r}(h) - C([r]).$$

□

Proposition 2.66. *The set $\{e_{i,k,r} \mid i \in \mathbb{Z}_{\geq 0}, 0 \leq k \leq r\}$ is a Banach basis for $\mathscr{C}^r(\mathbb{Z}_p, L)$.*

Proof. See [Col10]. □

Definition 2.67. *We define the valuation $\nu_{\mathscr{C}^r}$ on $\mathscr{C}^r(\mathbb{Z}_p, L)$ such that, for $\phi \in \mathscr{C}^r(\mathbb{Z}_p, L)$, writing ϕ in its Mahler expansion, $\phi(x) = \sum_{n\geq 0} a_n(\phi)\binom{x}{n}$, we have*

$$\nu_{\mathscr{C}^r}(\phi) = \inf_{n\in\mathbb{Z}_{\geq 0}}(\nu_p(a_n(\phi)) - rl(n)).$$

It can be checked that this valuation is equivalent to the valuation $\nu'_{\mathscr{C}^r}$.

Proposition 2.68. *If $r \geq 0$, $\exists C_1(r)$ such that for any $h \in \mathbb{Z}_{\geq 0}$ and $\phi \in LA_h(\mathbb{Z}_p, L)$, we have*

$$\nu_{\mathscr{C}^r}(\phi) \geq \nu_{LA_h}(\phi) - rh - C_1(r).$$

Proof. See [Col10]. □

2.7 Measures and Distributions on $\mathbb{Z}_p$

Definition 2.69. *Let $\mu : LA(\mathbb{Z}_p, L) \to L$. We say that μ is a locally analytic distribution on $\mathbb{Z}_p$ if μ is a continuous linear form on $LA(\mathbb{Z}_p, L)$.*

We define $\mathscr{D}(\mathbb{Z}_p, L)$ to be the set of locally analytic distributions on $\mathbb{Z}_p$.

If $\mu \in \mathscr{D}(\mathbb{Z}_p, L)$, we will write $\mu(f) = \int_{\mathbb{Z}_p} f(x)\mu(x)$ or more simply, $\mu(f) = \int_{\mathbb{Z}_p} f\mu$.

Definition 2.70. *Let $r \geq 0$, $\mu \in \mathscr{D}(\mathbb{Z}_p, L)$, then we say that μ is a distribution of order r if we can extend μ, to a continuous linear form $\mu : \mathscr{C}^r(\mathbb{Z}_p, L) \to L$.*

Let $\mathscr{D}_r(\mathbb{Z}_p, L)$ denote the set of distributions of order r.

We can think of the set of distributions of order r as the dual space of $\mathscr{C}^r(\mathbb{Z}_p, L)$. We can thus equip $\mathscr{D}_r(\mathbb{Z}_p, L)$ with the valuation $\nu_{\mathscr{D}_r}$, defined for $\mu \in \mathscr{D}_r(\mathbb{Z}_p, L)$, by

$$\nu_{\mathscr{D}_r}(\mu) = \inf_{f \in \mathscr{C}^r(\mathbb{Z}_p, L) \setminus \{0\}} \left(\nu_p \left(\int_{\mathbb{Z}_p} f\mu \right) - \nu_{\mathscr{C}^r}(f) \right).$$

Definition 2.71 (Measure)**.** *If $\mu \in \mathscr{D}_0(\mathbb{Z}_p, L)$, we say that μ is a measure.*

Proposition 2.72. *A distribution of order r is defined by its action on the binomial polynomial $\binom{x}{n}$.*

Proof. First note that for any $f \in \mathscr{C}^r(\mathbb{Z}_p, L)$, using Mahler's Theorem, we can write

$$f = \sum_{n=1}^{\infty} a_n(f) \binom{x}{n}.$$

Hence it is enough to show that

$$\int_{\mathbb{Z}_p} \left(\sum_{n=0}^{\infty} a_n \binom{x}{n} \right) \mu = \sum_{n=0}^{\infty} a_n \int_{\mathbb{Z}_p} \binom{x}{n} \mu.$$

Consider

$$\nu_p\left(\int_{\mathbb{Z}_p}\left(\sum_{n=0}^{\infty}a_n\binom{x}{n}\right)\mu-\sum_{n=0}^{k}a_n\int_{\mathbb{Z}_p}\binom{x}{n}\mu\right)=\nu_p\left(\int_{\mathbb{Z}_p}\left(\sum_{n=k+1}^{\infty}a_n\binom{x}{n}\right)\mu\right)$$
$$=\inf_{n\geq k+1}\nu_p\left(a_n\int_{\mathbb{Z}_p}\binom{x}{n}\mu\right).$$

Now since $\nu_p(a_n)\to\infty$, if we can show that $\nu_p\left(\int_{\mathbb{Z}_p}\binom{x}{n}\mu\right)$ is bounded from below, then we will have that $\nu_p(\int_{\mathbb{Z}_p}\left(\sum_{n=0}^{\infty}a_n\binom{x}{n}\right)\mu-\sum_{n=0}^{k}a_n\int_{\mathbb{Z}_p}\binom{x}{n}\mu)\to\infty$, and so we will have

$$\int_{\mathbb{Z}_p}\left(\sum_{n=0}^{\infty}a_n\binom{x}{n}\right)\mu=\sum_{n=0}^{\infty}a_n\int_{\mathbb{Z}_p}\binom{x}{n}\mu.$$

Now let $\sum_{j\geq 0}a_n\binom{x}{j}$ be the Mahler expansion of $\binom{x}{n}$, i.e $a_j=1$ if $j=n$ and is zero other wise.

Then

$$\nu_{\mathscr{C}^r}\left(\binom{x}{n}\right)=\inf_{j\in\mathbb{Z}_{\geq 0}}(\nu_p(a_j)-rl(j))=-rl(n)$$

since $\nu_p(a_j)=\infty$ unless $j=n$. Thus,

$$\nu_p\left(\int_{\mathbb{Z}_p}\binom{x}{n}\mu\right)\geq\nu_{\mathscr{D}_r}(\mu)+\nu_{\mathscr{C}^r}\left(\binom{x}{n}\right)\in L$$

and we have that $\nu_p\left(\int_{\mathbb{Z}_p}\binom{x}{n}\mu\right)$ is bounded from below. □

Definition 2.73. *We say that a p-adic measure μ is Haar if for every $\alpha\in\mathbb{Z}_p$, $\int_{\mathbb{Z}_p}\phi\mu=\int_{\mathbb{Z}_p}T_\alpha(\phi)\mu$ for all $\phi\in\mathscr{C}^0(\mathbb{Z}_p,L)$, where T_α is the translation map $T_\alpha:\mathscr{C}^0(\mathbb{Z}_p,L)\to\mathscr{C}^0(\mathbb{Z}_p,L)$ which satisfies $T_\alpha(\phi)=\phi(x+\alpha)$.*

Proposition 2.74. *There is no non-trivial p-adic Haar measure.*

Proof. Suppose that $\mu\in\mathscr{D}_0(\mathbb{Z}_p,L)$ is a p-adic Haar measure.

Using Lemma 2.40, we have that

$$\int_{\mathbb{Z}_p}\binom{x}{n}\mu=\int_{\mathbb{Z}_p}\binom{x-1}{n}\mu+\int_{\mathbb{Z}_p}\binom{x-1}{n-1}\mu.$$

Now since $-1\in\mathbb{Z}_p$ and μ is a Haar measure, we have

$$\int_{\mathbb{Z}_p}\binom{x}{n}\mu=\int_{\mathbb{Z}_p}\binom{x-1}{n}\mu$$

and hence $\int_{\mathbb{Z}_p} \binom{x-1}{n-1}\mu = 0, \forall\, n \geq 1$. Equivalently,

$$\int_{\mathbb{Z}_p} \binom{x}{n}\mu = 0$$

for all $n \geq 0$. Now let $\phi \in \mathscr{C}^0(\mathbb{Z}_p, L)$, then

$$\int_{\mathbb{Z}_p} \phi\mu = \sum_{n=0}^{\infty} a_n(\phi) \int_{\mathbb{Z}_p} \binom{x}{n}\mu = 0,$$

thus μ is a trivial measure. □

Definition 2.75. *Let I be a subset of $\mathbb{Z}_{\geq 0}$. We say that $\mu : LP^I(\mathbb{Z}_p, L) \to L$ is an algebraic distribution on $\mathbb{Z}_p$ if μ is a linear form on $LP^I(\mathbb{Z}_p, L)$.*

We denote $\mathscr{D}^I_{\texttt{alg}}(\mathbb{Z}_p, L)$ to be the set of algebraic distributions.

If $I \subset \mathbb{R}$, we let $\mathscr{D}^I_{\texttt{alg}}(\mathbb{Z}_p, L) = \mathscr{D}^{I\cap\mathbb{Z}_{\geq 0}}_{\texttt{alg}}(\mathbb{Z}_p, L)$.

Theorem 2.76. *i) If $\mu \in \mathscr{D}_r(\mathbb{Z}_p, L)$, $\exists\, c \in \mathbb{R}$ such that for any $a \in \mathbb{Z}_p$, $l \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 0}$, we have*

$$\nu_p\left(\int_{a+p^n\mathbb{Z}_p} \left(\frac{x-a}{p^n}\right)^l \mu\right) \geq c - rn.$$

ii) Conversely, if $N \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$, with $N \geq [r]$, and if $\mu \in \mathscr{D}^{[0,N]}_{\texttt{alg}}(\mathbb{Z}_p, L)$ with the property: $\exists\, c \in \mathbb{R}$ such that for any $a \in \mathbb{Z}_p$, $l \leq N$ and $n \in \mathbb{Z}_{\geq 0}$ we have

$$\nu_p\left(\int_{a+p^n\mathbb{Z}_p} \left(\frac{x-a}{p^n}\right)^l \mu\right) \geq c - rn.$$

Then μ extends uniquely to a distribution of order r on $\mathbb{Z}_p$.

Proof. This proof is taken from [Col10], with more detail added.

i) We define $\phi_{a,n,l} : \mathbb{Z}_p \to L$ such that

$$\phi_{a,n,l}(x) = \mathbb{1}_{a+p^n\mathbb{Z}_p}(x) \left(\frac{x-a}{p^n}\right)^l$$

for all $x \in \mathbb{Z}_p$; by definition we have that $\phi_{a,n,l} \in LA_n(\mathbb{Z}_p, L)$.

Now, for any $b \in \mathbb{Z}_p$, we write

$$\phi_{b,n}(x) = \begin{cases} 0 & \text{if } b \notin a + p^n\mathbb{Z}_p \\ \left(\frac{x-a}{p^n}\right) & \text{if } b \in a + p^n\mathbb{Z}_p. \end{cases}$$

Then, since $\nu_{LA_n}(\phi_{a,n,l}) = \inf_{b\in\mathbb{Z}_p} \nu_{b+p^h\mathbb{Z}_p}(\phi_{b,n})$ and

$$\nu_{b+p^h\mathbb{Z}_p}(\phi_{b,n}) = \begin{cases} \infty & \text{if } b \notin a + p^n\mathbb{Z}_p \\ 0 & \text{if } b \in a + p^n\mathbb{Z}_p, \end{cases}$$

we have that $\nu_{LA_n}(\phi_{a,n,l}) = 0$.

Now, since $\nu_{\mathscr{D}_r}(\mu) = \inf_{f\in\mathscr{C}^r(\mathbb{Z}_p,L)\setminus\{0\}}(\nu_p(\int_{\mathbb{Z}_p} f\mu) + \nu_{\mathscr{C}^r}(f))$, we have

$$\nu_p\left(\int_{a+p^n\mathbb{Z}_p}\left(\frac{x-a}{p^n}\right)^l \mu\right) \geq \nu_{\mathscr{D}_r}(\mu) + \nu_{\mathscr{C}^r}(\phi_{a,n,l}).$$

By Proposition 2.68, $\exists\, C_1(r)$ such that

$$\nu_{\mathscr{C}^r}(\phi_{a,n,l}) \geq \nu_{LA_n}(\phi_{a,n,l}) - rn - C_1(r) = -rn - C_1(r).$$

Hence

$$\nu_p\left(\int_{a+p^n\mathbb{Z}_p}\left(\frac{x-a}{p^n}\right)^l \mu\right) \geq \nu_{\mathscr{D}_r}(\mu) - rn - C_1(r).$$

Thus *i*) holds, with $c = \nu_{\mathscr{D}_r}(\mu) - C_1(r)$.

ii) We will first define

$$\nu_{\mathscr{D}_r,N}(\mu) = \inf_{a\in\mathbb{Z}_p, l\leq N, n\in\mathbb{Z}_{\geq 0}}\left(\nu_p\left(\int_{a+p^n\mathbb{Z}_p}\left(\frac{x-a}{p^n}\right)^l \mu\right) + rn\right)$$

for $\mu \in \mathscr{D}^{[0,N]}_{\mathtt{alg}}(\mathbb{Z}_p, L)$.

We now define $\mathscr{D}^{[0,N]}_r(\mathbb{Z}_p, L)$ to be the set of $\mu \in \mathscr{D}^{[0,N]}_{\mathtt{alg}}(\mathbb{Z}_p, L)$ for which $\exists\, A \in \mathbb{R}$ such that $\nu_{\mathscr{D}_r,N}(\mu) \geq A$.

Now suppose that $\mu \in \mathscr{D}(\mathbb{Z}_p, L)$, then μ is a continuous linear form on $LA(\mathbb{Z}_p, L)$. Since $LA(\mathbb{Z}_p, L) \subset LP^{[0,N]}(\mathbb{Z}_p, L)$, we have that $\mu \in \mathscr{D}^{[0,N]}_{\mathtt{alg}}(\mathbb{Z}_p, L)$, and hence there exists a natural map

$$\iota : \mathscr{D}(\mathbb{Z}_p, L) \to \mathscr{D}^{[0,N]}_{\mathtt{alg}}(\mathbb{Z}_p, L).$$

In particular, by part *i*), ι maps $\mathscr{D}_r(\mathbb{Z}_p, L)$ to $\mathscr{D}^{[0,N]}_r(\mathbb{Z}_p, L)$.

Thus, to prove *ii*), it is enough to show that $\iota\,|_{\mathscr{D}_r(\mathbb{Z}_p,L)}\colon \mathscr{D}_r(\mathbb{Z}_p, L) \to \mathscr{D}^{[0,N]}_r(\mathbb{Z}_p, L)$ is an isomorphism.

To prove injectivity of $\iota\,|_{\mathscr{D}_r(\mathbb{Z}_p,L)}$, we recall from Proposition 2.65 that $LP^{[0,N]}(\mathbb{Z}_p, L)$ is dense in $\mathscr{C}^r(\mathbb{Z}_p, L)$. Thus, if $\mu, \mu' \in \mathscr{D}_r(\mathbb{Z}_p, L)$ such that $\int_{\mathbb{Z}_p} \phi\iota(\mu) = \int_{\mathbb{Z}_p} \phi\iota(\mu')$ for all $\phi \in LP^{[0,N]}(\mathbb{Z}_p, L)$, then

$$\int_{\mathbb{Z}_p} \psi\mu = \int_{\mathbb{Z}_p} \psi\mu$$

for all $\psi \in \mathscr{C}^r(\mathbb{Z}_p, L)$. Thus $\mu = \mu'$.

It remains to prove surjectivity.

Let $\mu \in \mathscr{D}_r^{[0,N]}(\mathbb{Z}_p, L)$. For $i \in \mathbb{Z}_{\geq 0}$ and $l \in \{0, 1, \dots [r]\}$, let

$$b_{i,l} = \int_{\mathbb{Z}_p} e_{i,l,r}\mu = \int_{i+p^{l(i)}\mathbb{Z}_p} p^{[rl(i)]} \left(\frac{x - ai}{p^n}\right)^l \mu.$$

Then we have that $\nu_p(b_{i,l}) \geq \nu_{\mathscr{D}_r,N}(\mu) - 1$ for any $i \in \mathbb{Z}_{\geq 0}$ and $l \in \{0, 1, \dots [r]\}$.

By Proposition 2.66, $\{e_{i,l,r} \mid i \in \mathbb{Z}_{\geq 0}, 0 \leq l \leq [r]\}$ is a Banach basis for $\mathscr{C}^r(\mathbb{Z}_p, L)$. Thus we can uniquely define an element of $\mathscr{D}_r(\mathbb{Z}_p, L)$ by its action on the $e_{i,l,r}$. We then define a unique element $\tilde{\mu} \in \mathscr{D}_r(\mathbb{Z}_p, L)$ such that, for any $i \in \mathbb{Z}_{\geq 0}$ and $k \in \{0, 1, \dots [r]\}$,

$$\int_{\mathbb{Z}_p} e_{i,l,r}\tilde{\mu} = b_{i,l}.$$

Now let $\lambda := \iota(\tilde{\mu}) - \mu$, then by construction, this is an element of $\mathscr{D}_r^{[0,N]}(\mathbb{Z}_p, L)$. Also, for all $\phi \in LP^{[0,N]}(\mathbb{Z}_p, L)$,

$$\int_{\mathbb{Z}_p} \phi\lambda = 0.$$

We now note the following fact: if $a \in \mathbb{Z}_p$, $n, m \in \mathbb{Z}_{\geq 0}$, then

$$a + p^n\mathbb{Z}_p = a + p^n \left(\bigcup_{i=0}^{p^m-1} (i + p^m\mathbb{Z}_p) \right) = \bigcup_{i=0}^{p^m-1} a + ip^n + p^{n+m}\mathbb{Z}_p.$$

Thus, if $l \leq N$ and $m \in \mathbb{Z}_{\geq 0}$, we can write

$$\begin{aligned} \int_{a+p^n\mathbb{Z}_p} x^l\lambda &= \int_{a+p^n\mathbb{Z}_p} \left(\sum_{j=0}^{l} \binom{l}{j} (a + ip^n)^{l-j}(x - (a + ip^n))^j \right) \lambda \\ &= \sum_{i=0}^{p^m-1} \left(\sum_{j=0}^{l} \binom{l}{j} (a + ip^n)^{k-j} \int_{a+ip^n+p^{n+m}\mathbb{Z}_p} (x - (a + ip^n))^j\lambda \right). \end{aligned}$$

The terms in the above sum with $j \leq r$ are zero and, if $j > r$, then since

$$\nu_p \left(\int_{a+ip^n+p^{n+m}\mathbb{Z}_p} (x - (a + ip^n))^j\lambda \right) \geq \nu_{\mathscr{D}_r,N}(\lambda) + j(n + m) - r(n + m),$$

we have

$$\nu_p \left(\sum_{j=0}^{l} \binom{l}{j} (a + ip^n)^{k-j} \int_{a+ip^n+p^{n+m}\mathbb{Z}_p} (x - (a + ip^n))^j\lambda \right)$$

$$\geq \min_{r<j\leq l} \left(\nu_{\mathscr{D}_r,N}(\lambda) + (j-r)(n+m)\right).$$

Since the above tends to ∞ as $m \to \infty$, we deduce that

$$\int_{a+p^n\mathbb{Z}_p} x^l \lambda = 0$$

for all $a \in \mathbb{Z}_p$, $k \leq N$ and $n \in \mathbb{Z}_{\geq 0}$. Thus $\mu = \iota(\tilde{\mu})$ and so ι is surjective.

□

3 Modular Forms

The purpose of this section will be to develop the theory of modular forms that will allow us to define a distribution of order r on $\mathbb{Z}_p$ that takes values of interest on functions of the form x^n.

3.1 Definitions

Definition 3.1. *We define $\mathbb{H}$ to be the upper half plane in $\mathbb{C}$, $\mathbb{H} := \{x \in \mathbb{C} \mid \mathtt{im}(x) > 0\}$.*
Definition 3.2. *We let $SL_2(\mathbb{Z})$ denote the set of 2×2 matrices with determinant 1.*
Definition 3.3 (Congruence Subgroup)**.** *We first define*

$$\Gamma(N) := \left\{\gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\right\}.$$

Then we say that $\Gamma \subseteq SL_2(\mathbb{Z})$ is a congruence subgroup if $\exists\, N \geq 1$ such that $\Gamma(N) \subseteq \Gamma \subseteq SL_2(\mathbb{Z})$.

Proposition 3.4. $\Gamma_0(N) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N}\right\}$ *is a congruence subgroup.*

Proof. Clearly we have $\Gamma(N) \subseteq \Gamma_0(N) \subseteq SL_2(\mathbb{Z})$ and the proposition follows. □

Let $\{A_j \in SL_2(\mathbb{Z}) \mid j \in \mathscr{R}_N\}$ be coset representatives of $\Gamma_0(N)$, where $\mathscr{R}_N$ is a finite index set, such that $SL_2(\mathbb{Z}) = \sqcup_{j \in \mathscr{R}_N} \Gamma_0(N) A_j$.

If $\gamma \in GL_2(\mathbb{R})^+$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with positive determinant then we define, for $Z \in \mathbb{H}$, $\gamma z := \frac{az+b}{cz+d}$. We also define $\rho(\gamma)(z) = \frac{\det(\gamma)^{\frac{1}{2}}}{cz+d}$.

Definition 3.5. *A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k for a congruence subgroup Γ if* *i)* f *is holomorphic on* $\mathbb{H}$*;*

ii) $(cz+d)^{-k}f(\gamma z) = f(z)$, $\forall\, \gamma \in \Gamma$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\forall\, z \in \mathbb{H}$*;*

iii) $(cz+d)^{-k}f(\alpha z)$ *is holomorphic at* ∞, $\forall\, \alpha \in SL_2(\mathbb{Z})$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

A function f *is a cusp form of weight* k *for a congruence subgroup* Γ *if* f *satisfies i), ii), iii) above and in addition,* f *satisfies*

iv) $(cz+d)^{-k}f(\alpha z)$ *vanishes at* ∞, $\forall\, \alpha \in SL_2(\mathbb{Z})$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We now fix integers $k \geq 2$ even, $N \geq 1$. Then let $\mathscr{S}(N,k)$ denote the space of holomorphic cusp forms of weight k on $\Gamma_0(N)$.

Let $\mathscr{S}_k = \sum_{N\geq 1} \mathscr{S}(N,k)$ denote the space of all cusp forms of weight k which are on $\Gamma_0(N)$ for some $N \geq 1$.

Proposition 3.6 (Fourier Expansion of Modular Forms)**.** *Let* $f \in \mathscr{S}_k$*, then* $\exists\, a_n \in \mathbb{C}$ *such that*

$$f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}.$$

Proof. Taken from [DS06]. We first note that since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$ for all N, we have that $f(z+1) = f(z)$.

Now define the function $g : \mathbb{H} \to \mathbb{C}$ such that

$$g(q) = f\left(\frac{\log q}{2\pi i}\right).$$

In this definition, we are able to choose any branch of the logarithm by what we have noted above. Thus g is holomorphic on $\mathbb{H}$ and at infinity, so it has a Laurent expansion of the form

$$g(q) = \sum_{n=0}^{\infty} a_n q^n.$$

Finally, since f vanishes at infinity, so does g hence $a_0 = 0$ and thus we deduce

$$f(z) = g(e^{2\pi i z}) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

□

Let $\mathscr{P}_k(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C} \cdot z \oplus \cdots \oplus \mathbb{C} \cdot z^{k-2}$ denote the space of polynomials (in the variable z) of degree $\leq k-2$ with coefficients in $\mathbb{C}$.

Lastly, we define actions of $GL_2(\mathbb{Q})^+$ on $\mathscr{S}_k$ and on $\mathscr{P}_k(\mathbb{C})$ by the following formulas, letting $A \in GL_2(\mathbb{Q})^+$.

a) $(f \mid A)(z) := \rho(A)(z)^k f(Az)$ for $f \in \mathscr{S}_k$.

b) $(P \mid A)(z) := \rho(A)(z)^{2-k} P(Az)$ for $P \in \mathscr{P}_k(\mathbb{C})$.

We will soon see that the action on $\mathscr{P}_k(\mathbb{C})$ maps back into this space, however this is not true for the action on $\mathscr{S}_k$. To work with it, we will need a space which it maps into, so we make the following definition.

Definition 3.7. *We define a function $f : \mathbb{H} \to \mathbb{C}$ to be 'nice' if it satisfies the two properties:*

i) f is holomorphic;

ii) for all $\gamma \in GL_2(\mathbb{Q})^+$ there exists $C \in \mathbb{R}_{>0}$ such that as $z \to \infty$,

$$|f(\gamma z)| = O(e^{-C\mathtt{Im}(z)}).$$

i.e $\exists\, K > 0$ such that for all $z \in \mathbb{H}$ with $\mathtt{Im}(z) > K$, we have $|f(\gamma z)| = O(e^{-C\mathtt{Im}(z)})$.

We will denote these functions by $\mathscr{E}$.

Proposition 3.8. *Cusp forms of weight k on $\Gamma_0(N)$ are 'nice' functions, i.e $\mathscr{S}_k \subset \mathscr{E}$.*

Proof. First note that for $f \in \Gamma_0(N)$, f is holomorphic.

Now let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})^+$ and consider $\gamma z = \frac{az+b}{cz+d}$. If $c = 0$ then we write $\gamma z = x + iy$ for $x, y \in \mathbb{R}$. Then using the Fourier expansion for f, we have

$$\begin{aligned} |f(\gamma z)| &= \left| \sum_{n=1}^{\infty} a_n e^{2\pi i n(x+iy)} \right| \\ &\le \sum_{n=1}^{\infty} |a_n||e^{2\pi i n x}||e^{-2\pi n y}| \\ &\le \sum_{n=1}^{\infty} |a_n||e^{-2\pi n y}|. \end{aligned}$$

Since $f(z)$ vanishes at infinity $\exists\, K > 0$, such that for $\mathtt{Im}(z) > K$,

$$\sum_{n=1}^{\infty} |a_n||e^{-2\pi(n-1)y}|$$

is bounded. Thus there exists C, such that for all $\mathtt{Im}(z) > K$,

$$f(z) \le Ce^{-2\pi y}.$$

Now, if $c \neq 0$ we write $z = x + iy$ for $x, y \in \mathbb{R}$. Then we calculate

$$\frac{az+b}{cz+d} = \frac{(ax+b)(cx+d) + acy^2 + i(ad-bc)y}{(cx+d)^2 + c^2y^2}.$$

Thus

$$|f(\gamma z)| \leq \sum_{n=1}^{\infty} |a_n||e^{-2\pi n \frac{(ad-bc)}{(cx+d)^2+c^2y^2} y}|.$$

Now, let $K_1 > 0$ be such that $(cx+d)^2 + c^2y^2 > ad - bc$ for all $x \in \mathbb{R}$. Thus for all z such that $\mathtt{Im}(z) > K_1$ we have

$$|f(\gamma z)| \leq \sum_{n=1}^{\infty} |a_n||e^{-2\pi n y}|.$$

So, we are in the same case as before and the result follows. □

We can now characterise our actions on $\mathscr{S}_k$ and $P \in \mathscr{P}_k(\mathbb{C})$.

Proposition 3.9. *For $f \in \mathscr{S}_k$, $P \in \mathscr{P}_k(\mathbb{C})$ and $A \in GL_2(\mathbb{Q})^+$ we have the following:*

a) $(f \mid A)(z) \in \mathscr{E}$;

b) $(P \mid A)(z) \in \mathscr{P}_k(\mathbb{C})$.

Furthermore, if $f \in \mathscr{S}(N,k)$ and $A \in \Gamma_0(N)$ then $(f \mid A)(z) = f$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q})^+$.

a) Let $f \in \mathscr{S}_k$, so $\exists N \geq 1$ such that $f \in \mathscr{S}(N,k)$, letting $\Gamma = \Gamma_0(N)$, we check that $(f \mid A)(z)$ is a 'nice' function.

   i) Since $\rho(A)(z)$ and $\frac{az+b}{cz+d}$ are holomorphic on $\mathbb{H}$ then $(f \mid A)(z)$ is holomorphic on $\mathbb{H}$.

   ii) Let $\gamma \in GL_2(\mathbb{Q})^+$ then we can show by calculation that $f(\gamma(Az)) = f((\gamma A)z)$. Now since $\gamma A \in GL_2(\mathbb{Q})^+$ we have, as in Proposition 3.8, that there exists $K > 0$, such that for z with $\mathtt{Im}(z) > K$,

$$f(\gamma(Az)) = O(e^{-C\mathtt{Im}(z)})$$

   for some $C > 0$. Thus since $\rho(A)\left(\frac{az+b}{cz+d}\right)$ is bounded for z with $\mathtt{Im}(z) > K$ we have $(f|A)(\gamma z) = O(e^{-C\mathtt{Im}(z)})$.

Thus f is a 'nice' function.

b) Let $P \in \mathscr{P}_k$ and write $P = \sum_{n=0}^{k-2} a_n x^n$, then

$$(P \mid A)(z) = \frac{\mathtt{det}(A)^{\frac{2-k}{2}}}{(cz+d)^{2-k}} \sum_{n=0}^{k-2} a_n \left(\frac{az+b}{cz+d}\right)^n$$
$$= \mathtt{det}(A)^{\frac{2-k}{2}} \sum_{n=0}^{k-2} a_n (az+n)^n (cz+d)^{k-2-n} \in \mathscr{P}_k.$$

If $f \in \mathscr{S}(N,k)$ and $A \in \Gamma_0(N)$ then, since $\det(A) = 1$ and $f \in \mathscr{S}(N,k)$,

$$(f \mid A) = (cz+d)^{-k} f(Az) = f(z).$$

□

3.2 Modular Integrals

The remainder of this section is a special case of the work done in [Maz86], the theorems and proofs that follow are adapted from the corresponding theory in [Maz86].

Let $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$.

Definition 3.10 (Modular Integral)**.** *Let $k \geq 2$. We define $\phi : \mathscr{E} \times \mathscr{P}_k(\mathbb{C}) \times \mathbb{P}^1(\mathbb{Q}) \to \mathbb{C}$ by the formula*

$$\phi(f, P, r) = 2\pi i \int_{\infty}^{r} f(z)P(z)dz$$
$$= \begin{cases} 2\pi \int_0^\infty f(r+it)P(r+it)dt & \text{if } r \in \mathbb{Q} \\ 0 & \text{if } r = \infty. \end{cases}$$

When we want to consider functions with one of its variables fixed we shall show this in the equation by writing the fixed variable in a subscript.

Proposition 3.11. *The modular integral converges for all $(f, P, r) \in \mathscr{E} \times \mathscr{P}_k(\mathbb{C}) \times \mathbb{P}^1(\mathbb{Q})$.*

Proof. Let $f \in \mathscr{E}$ and $P \in \mathscr{P}_k(\mathbb{C})$. If $r = \infty$ then clearly the integral converges and is 0. Suppose now that $r \in \mathbb{Q}$.

Since $f \in \mathscr{E}$ there exists $K > 0$ such that for $z \in \mathbb{C}$ with $\mathtt{Im}(z) > K$, $|f(z)| = O(e^{-C\mathtt{Im}(z)})$ for some $C > 0$. Then we write

$$2\pi \int_0^\infty f(r+it)P(r+it)dt = 2\pi \int_0^K f(r+it)P(r+it)dt + 2\pi \int_K^\infty f(r+it)P(r+it)dt.$$

We will deal with the convergence of each of these integrals separately. To show that the second integral converges, we consider

$$\left|\int_K^\infty f(r+it)P(r+it)dt\right| \leq \int_K^\infty Ae^{-Ct}\left|P(r+it)\right| dt$$

where A is the constant such that $|f(z)| \leq Ae^{-C\mathtt{Im}(z)}$ for $\mathtt{Im}(z) > K$. Also note that since P is a polynomial and r is fixed, there exists a constant B such that $|P(r+it)| \leq B|t^{k-2}|$ for t greater then 1. Thus

$$\left|\int_K^\infty f(r+it)P(r+it)dt\right| \leq AB\int_K^\infty e^{-Ct}t^{k-2}dt.$$

Using integration by parts one can see that the integral on the RHS converges.

For the first integral we note that P is well defined for all t since it is a polynomial, however f is only well defined for $t > 0$. We will now consider the matrix

$$A := \begin{pmatrix} r & -r^2-1 \\ 1 & -r \end{pmatrix} \in GL_2(\mathbb{Q})^+$$

chosen such that $A(r+it) = r + i\frac{1}{t}$. Then

$$\int_0^K f(r+it)P(r+it)dt = -\int_{\frac{1}{K}}^\infty f(r+i\frac{1}{t})P(r+i\frac{1}{t})dt = -\int_{\frac{1}{K}}^\infty f(A(r+it))P(r+i\frac{1}{t})dt.$$

Now, since $f \in \mathscr{E}$ and $A \in GL_2(\mathbb{Q})^+$ there exists $K' > \frac{1}{K} > 0$, such that for z with $\mathtt{Im}(z) > K$, we have $f(Az) = O(e^{-C\mathtt{Im}(z)})$ for some $C > 0$. Hence

$$\left|\int_0^K f(r+it)P(r+it)dt\right|$$

$$\leq \left|\int_{\frac{1}{K}}^{K'} f(A(r+it))P(r+i\frac{1}{t})dt\right| + \left|\int_{K'}^\infty f(A(r+it))P(r+i\frac{1}{t})dt\right|.$$

Then similarly to the work we did for the second integral we can show that the first integral also converges. □

To prove properties of ϕ we first require the following lemma.

Lemma 3.12. *For $f \in \mathscr{E}$, $P \in \mathscr{P}_k(\mathbb{C})$ and $A \in GL_2(\mathbb{Q})^+$, we have*

$$(f \mid A)(z)(P \mid A)(z)dz = f(A(z))P(A(z))d(A(z)).$$

Proof. Consider $\frac{d(A(z))}{dz} = \frac{d}{dz}\left(\frac{az+b}{cz+d}\right) = \frac{a(cz+d)-c(az+b)}{(cz+d)^2} = \rho(A)(z)^2$.

Hence $dz = \rho(A)(z)^{-2}d(A(z))$.

Thus, we have

$$\begin{aligned}(f \mid A)(z)(P \mid A)(z) &= \rho(A)(z)^k f(Az)\rho(A)(z)^{2-k}P(Az)\rho(A)(z)^{-2} \\ &= f(A(z))P(A(z))\frac{d(A(z))}{dz}.\end{aligned}$$

□

Proposition 3.13. *ϕ satisfies the following*

i) $\phi(f,P,r)$ is $\mathbb{C}$-bilinear in f,P for any $r \in \mathbb{P}^1(\mathbb{Q})$;

ii) $\phi(f \mid A, P \mid A, r) = \phi(f,P,A(r)) - \phi(f,P,A(\infty))$ for any $A \in GL_2(Q)^+$.

Proof. We check the properties separately,

i) Let $f, g \in \mathscr{E}$, $\alpha, \beta \in \mathbb{C}$.

Then $\phi(\alpha f + \beta g, P, r) = 2\pi i \int_\infty^r (\alpha f + \beta g)(z)P(z)dz = \alpha\phi(f,P,r) + \beta\phi(g,P,r)$.

Similarly we can show ϕ is linear in P.

ii) Let $A \in GL_2(\mathbb{Q})^+$ and consider

$$\phi(f \mid A, P \mid A, r) = 2\pi i \int_\infty^r (f \mid A)(z)(P \mid A)(z)d(A(z)) = 2\pi i \int_{A(\infty)}^{A(r)} f(z)P(z)dz$$

using the lemma above. By Cauchy's Theorem we have

$$\int_{A(\infty)}^{A(r)} f(z)P(z)dz = \int_\infty^{A(r)} f(z)P(z)dz - \int_\infty^{A(\infty)} f(z)P(z)dz.$$

Hence $\phi(f \mid A, P \mid A, r) = \phi(f,P,A(r)) - \phi(f,P,A(\infty))$.

□

Definition 3.14. *Let $f \in \mathscr{S}_k$, V a complex vector space and ϕ our modular integral. We then define $L_f \subset V$ to be the $\mathbb{Z}$-module generated by the image of $\mathscr{P}_k(\mathbb{Z}) \times \mathbb{P}^1(\mathbb{Q})$ under the mapping ϕ_f.*

Proposition 3.15. *Let $f \in \mathscr{S}(N,k)$, V a complex vector space and ϕ a modular integral. Then the $\mathbb{Z}$-module L_f is the $\mathbb{Z}$-submodule of V generated by the elements*

$$X := \{\phi_f(z^i, A_j(\infty)) - \phi_f(z^i, A_j(0)) \mid 0 \le i \le k-2, j \in \mathscr{R}_N\}.$$

Proof. Let L be the $\mathbb{Z}$ module generated by the elements of X. Then clearly $L \subset L_f$.

Let $P \in \mathscr{S}_k$, $a, m \in \mathbb{Z}$ be relatively prime and $m \ge 0$, then we shall show that $\phi(f,P,\frac{a}{m}) \in L$ by induction on m.

If $m = 0$, then $\phi(f,P,\frac{a}{m}) = 0 \in L$.

Now suppose $m > 0$ and choose $m' \in \mathbb{Z}$ such that $am' \equiv 1 \mod m$ and $0 \leq m' < m$.

Now we let $a' = \frac{am'-1}{m}$ and $A = \begin{pmatrix} a & a' \\ m & m' \end{pmatrix} \in SL_2(\mathbb{Z})$.

Then $\exists j \in \mathscr{R}_N, B \in \Gamma_0(N)$ such that $A = BA_j$, thus we have

$$\begin{aligned}\phi(f, P, \frac{a}{m}) - \phi(f, P, \frac{a'}{m'}) &= \phi(f, P, A(\infty)) - \phi(f, P, A(0)) \\ &= \phi(f, P, BA_j(\infty)) - \phi(f, P, BA_j(0)) \\ &= \phi(f \mid B, P \mid B, A_j(\infty)) - \phi(f \mid B, P \mid B, A_j(0)).\end{aligned}$$

Note that, since $B \in \Gamma_0(N)$, we have that $f|B \in \mathscr{S}_k$. Thus, by Proposition 3.13, we have

$$\phi(f, P, \frac{a}{m}) - \phi(f, P, \frac{a'}{m'}) = \phi(f, P, A_j(\infty)) - \phi(f, P, A_j(0)).$$

Now, let $P = \sum_{n=0}^{k-2} a_n x^n$, then since ϕ is linear in $\mathscr{P}_k$ and by induction hypothesis $\phi(f, P, \frac{a'}{m'}) \in L$, we have that

$$\phi(f, P, \frac{a}{m}) = \phi(f, P, \frac{a'}{m'}) + \sum_{n=0}^{k-2} a_n(\phi(f, x^n, A_j(\infty) - \phi(f, x^n, A_j(0))).$$

□

3.3 Modular Symbols

Definition 3.16. *For $a, m \in \mathbb{Q}, m > 0, f \in \mathscr{S}_k$ and $P \in \mathscr{P}_k(\mathbb{C})$, we define*

$$\lambda(f, P; a, m) := \phi(f, P(mz + a), -\frac{a}{m}).$$

Proposition 3.17. *$\lambda(f, P; a, m)$ is $\mathbb{C}$-bilinear in (f, P), and fixing $f \in \mathscr{S}_k$, we have*

$$\lambda_f(P; a, m) \in L_f$$

for all $P \in \mathscr{P}_k(\mathbb{Z}), a, m \in \mathbb{Z},\ m > 0$.

Proof. $\mathbb{C}$-bilinaerity of λ follows from the $\mathbb{C}$-bilinaerity of ϕ.

By the definition of λ, it also follows that $\lambda_f(P; a, m) \in L_f$. □

Proposition 3.18. *If $a, m \in \mathbb{Z},\ m > 0$, then*

$$\lambda(f, (z - a)^i; a, m) \in m^i L_f.$$

Proof.

$$\lambda(f,(z-a)^i;a,m) = \phi(f,(mz)^i,-\frac{a}{m}) = m^i\phi(f,z^i,-\frac{a}{m}) \in m^i L_f.$$

□

3.4 Hecke Operators

The final piece of knowledge we require from modular forms is that of Hecke Operators.

Let $f \in \mathscr{S}(N,k)$, then for every prime number l, we define the operator T_l to be the following

$$f \mapsto f\,|_{T_l} := \begin{cases} l^{\frac{k}{2}-1}\left(\sum_{u=0}^{l-1} f \middle| \begin{pmatrix} 1 & u \\ 0 & l \end{pmatrix} + f \middle| \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}\right) & \text{if } l \nmid N \\ l^{\frac{k}{2}-1}\sum_{u=0}^{l-1} f \middle| \begin{pmatrix} 1 & u \\ 0 & l \end{pmatrix} & \text{if } l \mid N. \end{cases}$$

4 p-adic Distributions

4.1 p-adic Integrals

For this section, we fix a prime number p and $f \in \mathscr{S}(N,k)$ such that f is an eigenform of T_p with eigenvalue a_p. Let $v(k) = \texttt{ord}_p(k)$, $\forall\, k \in \mathbb{Z}$.

Now suppose also that the polynomial $X^2 - a_pX + p^{k-1}$ has a non-zero root in $\mathbb{R}$, let $\alpha \neq 0$ be such a root. We then define, for $P \in \mathscr{P}_k$, $a, z \in \mathbb{Z}$, $m > 0$,

$$\mu_{f,\alpha}(P;a,m) = \frac{1}{\alpha^{v(m)}}\lambda_{f,P}(a,m) - \frac{p^{k-2}}{\alpha^{v(m)+1}}\lambda_{f,P}(a,\frac{m}{p}).$$

Define the disc $D(a,n) := a + p^n\mathbb{Z}_p$ for $a \in \mathbb{Z}$ and $n \in \mathbb{Z}_{>0}$.

Let L be a finite field extension of $\mathbb{Q}_p$ that contains α. Let $\mathscr{O}_p := \{x \in L \mid \nu_p(x) \geq 0\} \subset L$ denote it's ring of integers, and $\mathscr{O}_p^* := \{x \in L \mid \nu_p(x) = 0\}$ its topological group of units. We fix L from now on.

We then define the finite dimensional L vector space $V_f := L \otimes_{\mathbb{Z}} L_f$ and the $\mathscr{O}_p$ lattice $\Omega_f := \{\sum_{i=1}^n x_iz_i \mid n \in \mathbb{Z}_{>0}, x_i \in L_f, z_i \in \mathscr{O}_p\} \subset V_f$.

Definition 4.1. *If $U \subset \mathbb{Z}_p$, we define a function $f : U \to L$ to be locally analytic if there exists a covering of U by discs $D(a,v)$ such that on each $D(a,v)$, F is given by the convergent power series $\sum_{n\geq 0} c_n(x-a)^n$.*

Note that the convergence of the above power series on $D(a, v)$ is equivalent to the condition that $\nu_p(p^{nv}c_n) = \nu_p(c_n) - nv \to \infty$.

We extend the definitions of $\phi(f, P, r)$, $\lambda(f, P; a, m)$ and $\mu_{f,\alpha}(P; a, m)$ to the case where P has coefficients in L giving values in V_f.

We define a V_f-valued integral to be a map $(U, F) \mapsto \int_U F \in V_f$, where U is any compact open subset of $\mathbb{Z}_p$ and F is a locally analytic function on U.

Theorem 4.2. *Suppose the polynomial $X^2 - a_pX + p^{k-1}$ has a root $\alpha \in L$ such that $\nu_p(\alpha) < k - 1$; fix such an α.*

Then there exists a unique V_f-valued integral satisfying the following axioms, where $n \geq 1$, $a \in \mathbb{Z}_p$.

1. *It is L_f-linear in F and finitely additive in U.*
2. *(Evaluation of polynomials of small degrees):*

   $\int_{D(a,n)} x^j = \mu_{f,\alpha}(z^i; a, p^n)$ *for* $0 \leq j < k - 1$.
3. *(Divisibility): For any $l \geq 0$,*

   $\int_{D(a,n)} (x - \rho(a))^l \in \left(\frac{p^l}{\alpha}\right)^n \alpha^{-1}\Omega_f$.
4. *(Continuity): If $F(x) = \sum_{l\geq 0} c_l(x - \rho(a))^l$ is convergent on the disc $D(a, v)$, then*

   $\int_{D(a,v)} F = \sum_{l\geq 0} c_l \int_{D(a,v)} (x - \rho(a))^l$.

Proof. Let $r := \nu_p(\alpha)$, then $k - 2 \geq [r]$. Using Theorem 2.76 it is enough to show the following two statements

i) $\int \in \mathscr{D}^{[0,k-2]}_{\mathtt{alg}}(\mathbb{Z}_p, L)$, and

ii) there exists $c \in \mathbb{R}$ such that

$$\nu_p\left(\int_{a+p^n\mathbb{Z}_p} \left(\frac{x-a}{p^n}\right)^l\right) \geq c - rn$$

for any $a \in \mathbb{Z}_p$, $l \leq k - 2$ and $n \in \mathbb{Z}_{\geq 0}$.

To show *i*), let $P, Q \in LP^{[0,k-2]}(\mathbb{Z}_p, L)$, $\gamma, \delta \in L$, then $\exists\, m \in \mathbb{Z}_{\geq 0}$ such that $P \mid_{a+p^m\mathbb{Z}_p}$ and $Q \mid_{a+p^m\mathbb{Z}_p}$ are polynomial of degree at most $k - 2$ for any $a \in \mathbb{Z}_p$, $m \in \mathbb{Z}_{\geq 0}$. We then write $P\mid_{a+p^m\mathbb{Z}_p} = \sum_{i=0}^{k-2} p_{a,m,i}z^i$ and $Q\mid_{a+p^m\mathbb{Z}_p} = \sum_{i=0}^{k-2} q_{a,m,i}z^i$. Now consider

$$\begin{aligned}\int_{\mathbb{Z}_p} \gamma P + \delta Q &= \sum_{a=0}^{p^m-1} \int_{D(a,m)} \gamma P + \delta Q \\ &= \sum_{a=0}^{p^m-1} \int_{D(a,m)} \sum_{i=0}^{h-1} (\gamma p_{a,m,i} + \delta q_{a,m,i}) z^i \\ &= \sum_{a=0}^{p^m-1} \sum_{i=0}^{k-2} (\gamma p_{a,m,i} + \delta q_{a,m,i}) \mu_{f,\alpha}(z^i; a, p^m) \\ &= \sum_{a=0}^{p^m-1} \left(\gamma \sum_{i=0}^{k-2} p_{a,m,i} \mu_{f,\alpha}(z^i; a, p^m) + \delta \sum_{i=0}^{k-2} q_{a,m,i} \mu_{f,\alpha}(z^i; a, p^m) \right) \\ &= \gamma \int_{\mathbb{Z}_p} P + \delta \int_{\mathbb{Z}_p} Q\end{aligned}$$

so $i)$ holds.

To prove $ii)$, let $a \in \mathbb{Z}_p$, $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq l \leq k-2$, then consider

$$\int_{D(a,n)} (x-a)^l = \int_{D(a',n)} (x - a' + a' - a)^l$$

where $a' \in \rho(\mathbb{Z}) \cap D(a,n)$. Thus

$$\int_{D(a,n)} (x-a)^l = \int_{D(a',n)} \sum_{j=0}^{l} \binom{l}{j} (x-a')^j (a'-a)^{l-j}.$$

Now since $\nu_p\left(\binom{l}{j}(a'-a)^{l-j}\right) \geq 0$, we deduce

$$\nu_p\left(\int_{D(a,n)} (x-a)^l\right) \geq \min_{j=0,\ldots,l} \left\{ \nu_p\left(\int_{D(a',n)} (x-a')^j\right) \right\}.$$

Thus, using 3) of Theorem 4.2, we have

$$\int_{D(a',n)} (x-a')^j \in \left(\frac{p^l}{\alpha}\right)^n \alpha^{-1}\Omega_f \implies \nu_p\left(\int_{D(a',n)} (x-a')^j\right) \geq nl - nr + \nu_p(\alpha^{-1}).$$

Hence

$$\nu_p\left(\int_{D(a,n)} \left(\frac{x-a}{p^n}\right)^l\right) \geq \nu_p(\alpha^{-1}) - rn$$

and so $ii)$ holds with $c = \nu_p(\alpha^{-1})$. $\square$

4.2 L-Functions

To end this section, we will prove a link between the p-adic integral we have defined and the L-function defined from a modular form. In this subsection, we will assume the following result on the Fourier coefficients of modular cusp forms.

Proposition 4.3. *Let $f \in \mathscr{S}_k$ then if a_n are the Fourier coefficients of f, we have*

$$a_n = O(n^{\frac{k}{2}}).$$

Proof. See [Miy89]. □

Definition 4.4 (Gamma Function)**.** $\Gamma : \mathbb{C} \to \mathbb{C}$ *is defined as*

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx.$$

Definition 4.5. *Let $f \in \mathscr{S}_k$, $f = \sum_{n \geq 1} a_n e^{2\pi i n z}$ we then define the L-function*

$$L(f, s) := \sum_{n \geq 1} a_n n^{-s}$$

for $s \in \{z \in \mathbb{C} \mid \mathtt{Im}(z) > \frac{k}{2} - 1\}$.

We note that by Corollary 4.3, the sum converges absolutely in the region of s given in the definition.

Proposition 4.6. *For $f \in \mathscr{S}_k$, $s \in \{z \in \mathbb{C} \mid \mathtt{Im}(z) > \frac{k}{2} - 1\}$ we have*

$$L(f, s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(it) t^s \left(\frac{dt}{t}\right).$$

Proof. We consider the integral

$$\int_0^\infty f(it) t^s \left(\frac{dt}{t}\right) = \int_0^\infty \left(\sum_{n \geq 1} a_n e^{2\pi i n (it)}\right) t^{s-1} dt.$$

Since the sum is absolutely convergent for our s we can interchange the sum and integral, thus

$$\int_0^\infty f(it) t^s \left(\frac{dt}{t}\right) = \sum_{n \geq 1} a_n \int_0^\infty e^{-2\pi n t} t^{s-1} dt.$$

Changing the integration variable to $x = 2\pi n t$, we deduce

$$\int_0^\infty f(it) t^s \left(\frac{dt}{t}\right) = \sum_{n \geq 1} \frac{a_n}{(2\pi n)^s} \int_0^\infty x^{s-1} e^{-x} dx = \frac{1}{(2\pi)^s} \Gamma(s) L(f, s).$$

The result follows. ∎

Proposition 4.7. *Let $f \in \mathscr{S}_k$, $0 \le n \le k-2$ and $n > \frac{k}{2}$, then*

$$\lambda(f, z^n; 0, 1) = i^n \frac{n!}{(2\pi)^n} L(f, n+1).$$

Proof. By the definition of λ, we have

$$\begin{aligned}
\lambda(f, z^n; 0, 1) &= \phi(f, z^n, 0) \\
&= 2\pi \int_0^\infty f(it)(it)^n dt \\
&= 2\pi i^n \int_0^\infty f(it) t^{n+1} \left(\frac{dt}{t}\right).
\end{aligned}$$

Now since $\Gamma(n+1) = n!$, we deduce

$$\begin{aligned}
\lambda(f, z^n; 0, 1) &= 2\pi i^n \frac{\Gamma(n+1)}{(2\pi)^{n+1}} \frac{(2\pi)^{n+1}}{\Gamma(n+1)} \int_0^\infty f(it) t^{n+1} \left(\frac{dt}{t}\right) \\
&= i^n \frac{n!}{(2\pi)^n} L(f, n+1).
\end{aligned}$$

∎

Proposition 4.8. *Let $f \in \mathscr{S}_k$, $0 \le n \le k-2$ and $n > \frac{k}{2}$, then*

$$\int_{\mathbb{Z}_p} x^n = \mu_{f,\alpha}(z^n; 0, 1) = \left(1 - \frac{p^{k-2}}{p^n \alpha}\right) i^n \frac{n!}{(2\pi)^n} L(f, n+1).$$

Proof. From [Maz86]. Using the definitions of μ and λ, we have

$$\begin{aligned}
\mu_{f,\alpha}(z^n; 0, 1) &= \lambda(f, z^n; 0, 1) - \frac{p^{k-2}}{\alpha} \lambda(f, z^n, 0, \frac{1}{p}) \\
&= \lambda(f, z^n; 0, 1) - \frac{p^{k-2}}{\alpha} \phi(f, \left(\frac{z}{p}\right)^n, 0) \\
&= \lambda(f, z^n; 0, 1) - \frac{p^{k-2}}{p^n \alpha} \phi(f, z^n, 0) \\
&= \left(1 - \frac{p^{k-2}}{p^n \alpha}\right) \lambda(f, z^n; 0, 1).
\end{aligned}$$

Thus, by Proposition 4.7, we deduce

$$\mu_{f,\alpha}(z^n; 0, 1) = \left(1 - \frac{p^{k-2}}{p^n \alpha}\right) i^n \frac{n!}{(2\pi)^n} L(f, n+1).$$

□

Although the above is a nice result, interpolating the value of x^n over $\mathbb{Z}_p$ is not always feasible, in particular if $x \in \mathbb{Z}_p$ is not a unit (for more detail on this see the second chapter of [Kob96]). However, interpolation works well over the units of $\mathbb{Z}_p$:

$$\mathbb{Z}_p^\star = \{x \in \mathbb{Z}_p \mid |x|_p = 1\}.$$

Thus the following proposition is of more interest.

Proposition 4.9. *Let $f \in \mathscr{S}_k$, $0 \leq n \leq k-2$ and $n > \frac{k}{2}$, then*

$$\int_{\mathbb{Z}_p^\star} x^n = \frac{1}{\alpha}\left(1 - \frac{p^{k-2-n}}{\alpha}\right)\left(1 - \frac{p^n}{\alpha}\right) i^n \frac{n!}{(2\pi)^n} L(f, n+1).$$

Proof. Since the p-adic integral is finitely additive in U, we have

$$\int_{\mathbb{Z}_p^\star} x^n = \int_{\mathbb{Z}_p} x^n - \int_{p\mathbb{Z}_p} x^n.$$

We have already calculated the first integral, so it remains to calculate the second integral. Since $p\mathbb{Z}_p = D(0,p)$, we have

$$\int_{p\mathbb{Z}_p} x^n = \int_{D(0,1)} x^n = \mu_{f,\alpha}(z^n; 0, p).$$

Using the definitions of $\mu_{f,\alpha}$, $\lambda_{f,P}$ and Proposition 4.7, we compute

$$\begin{aligned}\int_{p\mathbb{Z}_p} x^n &= \frac{1}{\alpha}\lambda_{f,x^n}(0,p) - \frac{p^{k-2}}{\alpha^2}\lambda_{f,x^n}(0,1)\\ &= \frac{p^n}{\alpha}\lambda_{f,x^n}(0,1) - \frac{p^{k-2}}{\alpha^2}\lambda_{f,x^n}(0,1)\\ &= \frac{1}{\alpha}\left(p^n - \frac{p^{k-2}}{\alpha}\right) i^n \frac{n!}{(2\pi)^n} L(f, n+1).\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathbb{Z}_p^\star} x^n &= \frac{1}{\alpha} i^n \frac{n!}{(2\pi)^n} L(f, n+1)\left(1 - \frac{p^{k-2-n}}{\alpha} - \frac{p^n}{\alpha} + \frac{p^{k-2}}{\alpha^2}\right)\\ &= \frac{1}{\alpha}\left(1 - \frac{p^{k-2-n}}{\alpha}\right)\left(1 - \frac{p^n}{\alpha}\right) i^n \frac{n!}{(2\pi)^n} L(f, n+1).\end{aligned}$$

□

Using analytic continuation, it is possible to extend our definition of $L(f, s)$ to the whole of the complex plane, and then in Propositions 4.8, 4.7 and 4.9 we are able to drop the condition that $n > k/2$.

5 Conclusion

In this report we have developed the theory of p-adic functions and distributions in order to consider, using modular forms, a p-adic distribution on $\mathscr{C}^r$. In the closing section of this project, we showed that the distribution we developed is very closely linked to the L-function attached to our chosen modular form. In particular we showed the relation between the integral of functions of the form x^n with $0 \leq n \leq k-2$ and $n > \frac{k}{2}$ over $\mathbb{Z}_p^\star$. As we stated before, the reason for considering this integral rather than over $\mathbb{Z}_p$ is that using interpolation, we can extend this to a result for all n. Thus we are able to obtain more information about L-functions. To further this work we can first define the 'twist' of a modular form which we obtain by adjusting the Fourier expansion of a modular form by multiplying each term by $\chi(n)$, where χ is a Dirichlet character (for more information on Dirichlet characters see chapter 3 of [Was97]). This can allow us to show the non-vanishing of L-functions at all but finitely many points of the central values of twists.

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