MA4H9 Modular Forms: Problem Sheet 1

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This is the first of 3 problem sheets, each of which amounts to 5% of your final mark for the course. This problem sheet will be marked out of a total of 40; the number of marks available for each question is indicated. See the end of the sheet for some formulae that you may quote without proof.

You should hand in your work to the Undergraduate Office by ***3pm on Friday 12th November****.*

1. [6 points] Let $\mathrm{GL}_2(\mathbb{R})$ be the group of 2×2 invertible real matrices, and $\mathrm{GL}_2^+(\mathbb{R})$ the subgroup of matrices with positive determinant.
   (a) Show that the formula
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = \frac{az+b}{cz+d},$$
   gives an action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{C} \setminus \mathbb{R}$.
   (b) Show that the stabiliser in $\mathrm{GL}_2(\mathbb{R})$ of $\mathcal{H}$ is $\mathrm{GL}_2^+(\mathbb{R})$.
   (c) Show that $g \in \mathrm{GL}_2^+(\mathbb{R})$ maps i to itself if and only if g is of the form $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ for $x, y \in \mathbb{R}$ (not both zero). Give a corresponding characterisation of the elements which send i to $-i$.

2. [2 points] Show that if f, f' are functions on $\mathcal{H}$, $k, k' \in \mathbb{Z}$, and $g \in \mathrm{SL}_2(\mathbb{R})$, then $(f|_k g)(f'|_{k'} g) = (ff')|_{k+k'} g$.

3. [6 points] Let $\lambda \in \mathbb{R}$ with $0 < \lambda < 2$, and let $\Gamma_\lambda \subseteq \mathrm{SL}_2(\mathbb{R})$ be the group generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.
   (a) Show that every Γ_λ-orbit in $\mathcal{H}$ contains a point in the set
   $$\mathcal{D}_\lambda = \{z : |z| \ge 1, -\tfrac{\lambda}{2} \le \operatorname{Re} z \le \tfrac{\lambda}{2}\}.$$
   (b) Find a point $z \in \mathcal{D}_\lambda$ whose stabiliser in Γ_λ has order > 4.
   (c) Show that if every point of $\mathcal{H}$ has finite stabiliser in Γ_λ, then there exists $t \in \mathbb{Q}$ such that $\lambda = 2\cos t\pi$.

4. [4 points] (a) Let f and g be modular forms of the same weight k. Show that $F(z) = f(z)\overline{g(z)}(\operatorname{Im} z)^k$ satisfies $F(\gamma z) = F(z)$ for all $\gamma \in G$.
   (b) Hence (or otherwise) show that if f is a cuspidal modular form of weight k, then $|f(z)|\,(\operatorname{Im} z)^{k/2}$ is bounded on $\mathcal{H}$.

5. [2 points] Show that $\sigma_t(mn) = \sigma_t(m)\sigma_t(n)$ for m, n coprime integers.

6. [4 points] Show that for any $z \in \mathcal{H}$ which is not in the G-orbit of ρ or i, there exists a nonzero $f \in M_{12}$ such that $f(z) = 0$; this f is unique up to scaling; and it has a simple zero at every point in the G-orbit of z and no other zeroes in $\mathcal{H}$. What is the corresponding statement for the two "bad" orbits?

7. [6 points] Show that the q-expansion coefficients of Δ lie in $\mathbb{Z}$, where we define $\Delta = (E_4^3 - E_6^2)/1728$. *(You may* **not** *use the product formula for* Δ *in this question.)*

8. [6 points] Recall that for any modular function f, $a_i(f)$ denotes the coefficient of q^i in the q-expansion of f.
   (a) Show that if $f \in M_{16}$ satisfies $a_1(f) = a_2(f) = 0$, then $f = 0$.
   (b) Find constants λ_1, λ_2 such that $g = \lambda_1 E_4^4 + \lambda_2 E_4 \Delta$ satisfies $a_1(g) = 1$ and $a_2(g) = \sigma_{15}(2)$.
   (c) Hence calculate the constant γ_{16} such that $E_{16} = 1 + \gamma_{16} \sum_{n \geq 1} \sigma_{15}(n) q^n$.

9. [4 points] Find constants λ_1, λ_2 such that $E_4^3 = \lambda E_{12} + \mu \Delta$. Hence prove "Ramanujan's congruence": if $\tau(n) = a_n(\Delta)$, then $\tau(n) = \sigma_{11}(n) \pmod{691}$.

10. (*For amusement only – not assessed*) Let H be the abstract group generated by two elements a, b such that $a^2 = b^3 = 1$, with no other relations.
   (a) *[Easy]* Show that there is a well-defined surjective map $H \to G$ mapping a to S and b to ST.
   (b) *[Hard]* Show that this map is an isomorphism. (Hint: Consider the elements $x = T$ and $y = TST = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in Γ. Show by induction that any nonempty product of the form $x^{m_1} y^{n_1} \dots x^{m_r} y^{n_r}$ with $m_i, n_i \geq 0$ must have at least one off-diagonal entry positive.)

In any of the above questions you may use without proof the following q-expansion formulae:

$$E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n$$
$$E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n$$
$$E_{12} = 1 + \tfrac{65520}{691} \sum_{n \geq 1} \sigma_{11}(n) q^n$$
$$\Delta = q - 24q^2 + O(q^2)$$