MA4H9 Modular Forms: Problem Sheet 2

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This is the second of 3 problem sheets, each of which amounts to 5% of your final mark for the course. This problem sheet will be marked out of a total of 40; the number of marks available for each question is indicated. See the end of the sheet for some formulae that you may quote without proof.

You should hand in your work to the Undergraduate Office by ***3pm on Friday 3rd December****.*

1. [4 points] Let p be prime and $\tau \in \mathcal{H}$. Prove that the subgroups of $\mathbb{Z} + \mathbb{Z}\tau$ of index p are $p\mathbb{Z} + (\tau + j)\mathbb{Z}$, for $j = 0, \dots, p-1$, and $\mathbb{Z} + p\tau\mathbb{Z}$. Show that there are $p^2 + p + 1$ subgroups of index p^2, and give a list of these.

2. [4 points] Let p be prime and $j \ge 1$, and suppose $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ has q-expansion $\sum_{n \ge 0} a_n q^n$. Give a proof of the formula in Lemma 1.6.10,
$$T_{p^j}(f) = \left(\sum_{n\ge 0} a_{np^j} q^n\right) + p^{k-1}\left(\sum_{n \ge 0} a_{np^{j-1}} q^{np}\right) + \cdots + p^{j(k-1)}\left(\sum_{n\ge 0} a_n q^{p^j n}\right).$$
(Hint: Consider the operators U, V on the ring $\mathbb{C}[[q]]$ of formal power series defined by $U\left(\sum a_n q^n\right) = \sum a_{np} q^n$, $V\left(\sum a_n q^n\right) = p^{k-1} \sum a_n q^{np}$.)

3. [5 points] Let f be a normalised eigenform in $M_k(\mathrm{SL}_2(\mathbb{Z}))$ and p a prime. Let α and β be the roots of the polynomial $X^2 - a_p(f)X + p^{k-1}$.
   (a) Show that $a_{p^r}(f) = \alpha^r + \alpha^{r-1}\beta + \cdots + \alpha\beta^{r-1} + \beta^r$ for all $r \ge 0$.
   (b) Show that if $|a_p(f)| \le 2p^{(k-1)/2}$, then $|\alpha| = |\beta| = p^{(k-1)/2}$.
   (c) Show that if the hypothesis of part (b) holds, then $a_{p^r}(f) \le (r+1)p^{r(k-1)/2}$ for all $r \ge 0$.
   (d) Deduce that if the hypothesis of part (b) holds for all primes p, then $a_n(f) \le d(n) n^{(k-1)/2}$ for all $n \in \mathbb{N}$, where $d(n) = \sigma_0(n)$ is the number of divisors of n.

4. [4 points] Calculate the matrix of the Hecke operator T_2 acting on $S_{32}(\mathrm{SL}_2(\mathbb{Z}))$ (in a basis of your choice). Show that its characteristic polynomial is $x^2 - 39960x - 2235350016$. *(Hint: Use a computer to do the algebra!)*

5. [5 points] Let $N \ge 2$ and let $c, d \in \mathbb{Z}/N\mathbb{Z}$. We say that c and d are *coprime modulo N* if there is no $f \neq 0$ in $\mathbb{Z}/N\mathbb{Z}$ such that $fc = fd = 0$.
   (a) Show that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, then c and d are coprime modulo N.
   (b) Show that for any pair (c, d) that are coprime modulo N, there exist $c', d' \in \mathbb{Z}$ such that $c' = c$ and $d' = d \pmod N$ and $\mathrm{HCF}(c', d') = 1$.
   (c) Hence (or otherwise) show that the natural reduction map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective for any $n \ge 2$.
   (d) Give an example of an integer N and an element of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ which is not in the image of $\mathrm{GL}_2(\mathbb{Z})$.

6. [4 points] The *Sanov subgroup* of $\mathrm{SL}_2(\mathbb{Z})$ is the set S of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a = d = 1 \pmod 4$ and $b = c = 0 \pmod 2$.

(a) Show that S is indeed a subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

(b) Show that S is a congruence subgroup, and determine its level.

(c) Show that S has index 12 in $\mathrm{SL}_2(\mathbb{Z})$.

7. [1 point] Show that $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ for any $N \geq 1$.

8. [3 points] Let Γ be an odd subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (that is, $-1 \neq \Gamma$).

   (a) Show that the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ is even.

   (b) Show that there is no odd subgroup of index 2.

9. [3 points] Let f be a modular function of level $\mathrm{SL}_2(\mathbb{Z})$ (and some weight k) and let p be prime. Show that $f(pz)$ is a modular function of level $\Gamma_0(p)$, and calculate $v_{\Gamma_0(p),c}(f(pz))$ for the two cusps $c \in C(\Gamma_0(p))$. Hence show that $f(pz)$ is a modular form or cusp form if and only if f is.

10. [2 points] Let $p \geq 3$ be prime. Show that for each cusp $c \in C(\Gamma_0(p))$, there are $\frac{p+1}{2}$ distinct cusps in $C(\Gamma_1(p))$ which are equivalent to c in $C(\Gamma_0(p))$.

11. [2 points] Show that $1/2$ is an irregular cusp of $\Gamma_1(4)$, and calculate its width.

12. [3 points] Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, and let $g \in \mathrm{SL}_2(\mathbb{Z})$. Show that gi has nontrivial stabiliser in $\overline{\Gamma}$ if and only if $\pm g \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} g^{-1} \in \overline{\Gamma}$. Hence show that there exist points $z \in \mathcal{H}$ with $n_{\Gamma_0(N)}(z) = 2$ if and only if -1 is a square modulo N.

13. *(Non-assessed and for amusement only – I don't know the answer to this one)* Does there exist a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ for which every cusp is irregular?

In any of the above questions you may use without proof the following q-expansion formulae:

$$E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n$$
$$\Delta = q - 24q^2 + 252q^3 - 1472q^4 + O(q^5)$$