MA4H9 Modular Forms: Problem Sheet 1 – Solutions

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1. [6 points] Let $\mathrm{GL}_2(\mathbb{R})$ be the group of 2×2 invertible real matrices, and $\mathrm{GL}_2^+(\mathbb{R})$ the subgroup of matrices with positive determinant.

   (a) Show that the formula
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z = \frac{az+b}{cz+d},$$
   gives an action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{C} \setminus \mathbb{R}$.

   Solution: Recall that in the course of proving proposition 1.1.2 we derived the identity
$$\mathrm{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z \right) = \frac{ad-bc}{|cz+d|^2} \mathrm{Im}(z).$$
   This is valid for any $a, b, c, d \in \mathbb{R}$ and $z \in \mathbb{C}$ with $cz + d \ne 0$, and shows that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ z \notin \mathbb{R}$ if $z \notin \mathbb{R}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. The proof we gave that $g_1 \circ (g_2 \circ z) = (g_1 g_2) \circ z$ is still valid in this generality, so we obtain a well-defined action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{C} \setminus \mathbb{R}$.

   (b) Show that the stabiliser in $\mathrm{GL}_2(\mathbb{R})$ of $\mathcal{H}$ is $\mathrm{GL}_2^+(\mathbb{R})$.

   Solution: From the above identity it's clear that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ preserves $\mathcal{H}$ and $\overline{\mathcal{H}}$ if $ad - bc > 0$, and swaps them around if $ad - bc < 0$, so the stabiliser of $\mathcal{H}$ is $\mathrm{GL}_2^+(\mathbb{R})$.

   (c) Show that $g \in \mathrm{GL}_2^+(\mathbb{R})$ maps i to itself if and only if g is of the form $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ for $x, y \in \mathbb{R}$ (not both zero). Give a corresponding characterisation of the elements which send i to $-i$.

   Solution: Straightforward algebra shows that $\frac{ai+b}{ci+d} = i$ if and only if $ad - bc = c^2 + d^2$ and $bd + ac = 0$. For each choice of sign, this is a pair of simultaneous linear equations for a, b (regarding c, d as fixed); the determinant is $c^2 + d^2 \ne 0$, and the unique solution is $a = d, b = -c$. Similarly, one shows that $g \circ i = -i$ if and only if g is of the form $\begin{pmatrix} x & y \\ y & -x \end{pmatrix}$.

2. [2 points] Show that if f, f' are functions on $\mathcal{H}$, $k, k' \in \mathbb{Z}$, and $g \in \mathrm{SL}_2(\mathbb{R})$, then $(f|_k g)(f'|_{k'} g) = (ff')|_{k+k'} g$.

   Solution: Recall that the weight k action is defined by $(f|_k g)(z) = j(g,z)^{-k} f(g \circ z)$. So we have
$$\begin{aligned}(f|_k g)(z)(f'|_{k'} g)(z) &= j(g,z)^{-k} f(g \circ z) j(g,z)^{-k'} f'(g \circ z) \\ &= j(g,z)^{-(k+k')} f(g \circ z) f'(g \circ z) = (ff'|_{k+k'} g)(z).\end{aligned}$$

3. [6 points] Let $\lambda \in \mathbb{R}$ with $0 < \lambda < 2$, and let $\Gamma_\lambda \subseteq \mathrm{SL}_2(\mathbb{R})$ be the group generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.

   (a) Show that every Γ_λ-orbit in $\mathcal{H}$ contains a point in the set
   $$\mathcal{D}_\lambda = \{z : |z| \geq 1, -\tfrac{\lambda}{2} \leq \operatorname{Re} z \leq \tfrac{\lambda}{2}\}.$$

   (b) Find a point $z \in \mathcal{D}_\lambda$ whose stabiliser in Γ_λ has order > 4.

   (c) Show that if every point of $\mathcal{H}$ has finite stabiliser in Γ_λ, then there exists $t \in \mathbb{Q}$ such that $\lambda = 2\cos t\pi$.

> **Solution:** The first part follows exactly as in the special case $\lambda = 1$, which we covered in lectures.
>
> Let $\theta \in (0, \frac{\pi}{2})$ be such that $\lambda = 2\cos\theta$, and consider $z = e^{i\theta}$. This is a "corner" of $\mathcal{D}_\lambda$. One checks that
> $$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^{-1} z = \frac{-1}{e^{i\theta} - 2\cos\theta} = e^{i\theta} = z.$$
> So this group element stabilises z, and its order is clearly > 4 (since $\theta < \pi/2$).
>
> One checks that the derivative of $z \mapsto \frac{-1}{z - 2\cos\theta}$ at $z = e^{i\theta}$ is $e^{i\theta}$. If the stabiliser of z is finite, this had better be a root of unity of finite order, so θ is a rational multiple of π.

4. [4 points] (a) Let f and g be modular forms of the same weight k. Show that $F(z) = f(z)\overline{g(z)}(\operatorname{Im} z)^k$ satisfies $F(\gamma z) = F(z)$ for all $\gamma \in G$.

> **Solution:** One checks that
> $$\begin{aligned} F(\gamma z) &= f(\gamma z)\overline{g(\gamma z)}(\operatorname{Im} \gamma z)^k \\ &= \left(j(\gamma, z)^k f(z)\right) \cdot \overline{\left(j(\gamma, z)^k g(z)\right)} \cdot \left(\frac{\operatorname{Im} z}{|j(\gamma, z)|^2}\right)^k \\ &= \frac{j(\gamma, z)^k \overline{j(\gamma, z)^k}}{|j(\gamma, z)|^{2k}} \cdot F(z) \\ &= F(z). \end{aligned}$$

   (b) Hence (or otherwise) show that if f is a cuspidal modular form of weight k, then $|f(z)|\,(\operatorname{Im} z)^{k/2}$ is bounded on $\mathcal{H}$.

> **Solution:** If f is cuspidal, then $F(z) = |f(z)|\,(\operatorname{Im} z)^{k/2}$ is a continuous function on $\mathcal{D}$. It tends to 0 as $\operatorname{Im} z \to \infty$ (because f is cuspidal); so it is bounded on $\mathcal{D}$. But it is also G-invariant, by part (a). Since every G-orbit in $\mathcal{H}$ contains a point of $\mathcal{D}$, it follows that any upper bound for $F(z)$ on $\mathcal{D}$ is actually an upper bound for F on $\mathcal{H}$.

5. [2 points] Show that $\sigma_t(mn) = \sigma_t(m)\sigma_t(n)$ for m, n coprime integers.

> **Solution:** Since m and n are coprime, every divisor d of mn can be uniquely written in the form $d_1 d_2$ where $d_1 \mid m$ and $d_2 \mid n$. Hence
> $$\sum_{d \mid mn} d^t = \sum_{d_1 \mid m, d_2 \mid n} d_1^t d_2^t = \left(\sum_{d_1 \mid m} d_1^t\right)\left(\sum_{d_2 \mid n} d_2^t\right).$$

6. [4 points] Show that for any $z \in \mathcal{H}$ which is not in the G-orbit of ρ or i, there exists a nonzero $f \in M_{12}$ such that $f(z) = 0$; this f is unique up to scaling; and it has a simple zero at every point in the G-orbit of z and no other zeroes in $\mathcal{H}$. What is the corresponding statement for the two "bad" orbits?

> **Solution:** We know that Δ has no zeroes in $\mathcal{H}$. Hence $\Delta(z) \neq 0$. So there is a unique λ for which $E_{12} - \lambda\Delta$ vanishes at z. By the valence formula, this must be a simple zero, and there can be no other zeroes. If there were two linearly independent functions in M_{12} vanishing at z, then we could take a suitable linear combination of them to find a function vanishing to order 2 at z, which is impossible.
>
> If $z = i$ or $z = \rho$, then there exists a nonzero $f \in M_{12}$ vanishing at z, for exactly the same reason. It's clear from the valence formula that any such f vanishes at z to order exactly 2 (if $z = i$) or 3 (if $z = \rho$), and has no other zeroes. As before, it must be unique up to scaling because otherwise we could take a linear combination to construct a nonzero form vanishing to higher order at z.

7. [6 points] Show that the q-expansion coefficients of Δ lie in $\mathbb{Z}$, where we define $\Delta = (E_4^3 - E_6^2)/1728$. *(You may* **not** *use the product formula for Δ in this question.)*

> **Solution:** (I love this question.) It suffices to show that E_4^3 and E_6^2 are congruent modulo 1728, where we say that two power series in q with integer coefficients are congruent if all of their coefficients are.
>
> We have $E_4 = 1 + 240 \sum_n \sigma_3(n) q^n$. When we cube this we get
>
> $$E_4^3 = 1 + 720\left(\sum_n \sigma_3(n)q^n\right) + 172800\left(\sum_n \sigma_3(n)q^n\right)^2 + 13824000\left(\sum_n \sigma_3(n)q^n\right)^3.$$
>
> The coefficients 172800 and 13824000 are divisible by 1728, so we conclude that
>
> $$E_4^3 = 1 + 720\sum_n \sigma_3(n)q^n \pmod{1728}.$$
>
> Similarly,
>
> $$E_6^2 = 1 - 1008\left(\sum_n \sigma_5(n)q^n\right) + 254016\left(\sum_n \sigma_5(n)q^n\right)^2.$$
>
> Clearly $-1008 = 720 \pmod{1728}$, and 254016 is a multiple of 1728, so
>
> $$E_6^2 = 1 + 720\sum_n \sigma_5(n)q^n \pmod{1728}.$$
>
> So we need to check that $720\sigma_3(n) = 720\sigma_5(n) \pmod{1728}$. Cancelling out the greatest common factor of 720 and 1728 (which is 144), this is equivalent to $\sigma_3(n) = \sigma_5(n) \pmod{12}$. But it's trivial to check that $d^3 = d^5 \bmod 12$ for all $d \in \mathbb{N}$, so summing over the divisors of n gives the result.

8. [6 points] Recall that for any modular function f, $a_i(f)$ denotes the coefficient of q^i in the q-expansion of f.

   (a) Show that if $f \in M_{16}$ satisfies $a_1(f) = a_2(f) = 0$, then $f = 0$.

   > **Solution:** The space M_{16} is spanned by E_4^4 and $E_4\Delta$. Using the formulae for E_4 and Δ given on the sheet, we calculate that the q-expansions (to order 2) of these are $1 + 960q +$

$354240q^2 + \dots$ and $q + 216q^2 + \dots$. Since the determinant $\begin{vmatrix} 960 & 354240 \\ 1 & 216 \end{vmatrix} = -146880 \neq 0$, there is no non-vanishing linear combination of these two forms with $a_1 = a_2 = 0$.

(b) Find constants λ_1, λ_2 such that $g = \lambda_1 E_4^4 + \lambda_2 E_4 \Delta$ satisfies $a_1(g) = 1$ and $a_2(g) = \sigma_{15}(2)$.

Solution: We need to solve the matrix equation

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} 960 & 354240 \\ 1 & 216 \end{pmatrix} = \begin{pmatrix} 1 & 32769 \end{pmatrix}.$$

The unique solution is $\lambda_1 = 3617/16320, \lambda_2 = -3600/17$.

(c) Hence calculate the constant γ_{16} such that $E_{16} = 1 + \gamma_{16} \sum_{n \geq 1} \sigma_{15}(n) q^n$.

Solution: We know that $\gamma_{16}^{-1} E_{16} = \gamma_{16}^{-1} + q + \sigma_{15}(2)q^2 + \dots$, and $\lambda_1 E_4^4 + \lambda_2 E_4 \Delta$ has the same coefficient of q and q^2. So it must have the same constant term as well, by part (a). We deduce that $\gamma_{16} = \lambda_1^{-1} = \frac{16320}{3617}$.

9. [4 points] Find constants λ_1, λ_2 such that $E_4^3 = \lambda E_{12} + \mu \Delta$. Hence prove "Ramanujan's congruence": if $\tau(n) = a_n(\Delta)$, then $\tau(n) = \sigma_{11}(n) \pmod{691}$.

Solution: We'd better have $\lambda = 1$, otherwise the constant terms won't match. Comparing the linear terms, we find that $\mu = \frac{432000}{691}$. Thus the coefficient of q^n in E_4^3 must be $65520/691\sigma_{11}(n) + \frac{432000}{691}\tau(n)$. But E_4^3 has coefficients in $\mathbb{Z}$, so

$$432000\tau(n) = -65520\sigma_{11}(n) \pmod{691}.$$

Since $432000 = -65520 = 125 \pmod{691}$, and 125 is obviously coprime to 691, we conclude that $\tau(n) = \sigma_{11}(n) \pmod{691}$.

10. (*For amusement only – not assessed*) Let H be the abstract group generated by two elements a, b such that $a^2 = b^3 = 1$, with no other relations.

(a) *[Easy]* Show that there is a well-defined surjective map $H \to G$ mapping a to S and b to ST.

Solution: It is easy to see that $S^2 = (ST)^3 = 1$ in G, so the map is well-defined. Moreover, since S and T generate G, so do S and ST; thus the map is surjective.

(b) *[Hard]* Show that this map is an isomorphism. (Hint: Consider the elements $x = T$ and $y = TST = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in Γ. Show by induction that any nonempty product of the form $x^{m_1} y^{n_1} \dots x^{m_r} y^{n_r}$ with $m_i, n_i \geq 0$ must have at least one off-diagonal entry positive.)

Solution: The lemma suggested in the hint follows by induction on the length of the product, since any such product has entries which are non-negative integers (so no "cancellation" occurs in the matrix multiplication). Hence no such product reduces to the identity, except the empty product.

Let z be an element of H mapping to the identity. We may consider z as a word in a and b, of the form

$$z = b^r a b^{e_1} a b^{e_2} \dots a b^{e_n} a^s$$

where $r = 0, 1$ or 2, $s = 0$ or 1, and each e_i is 0 or 1.

By conjugating by a if necessary, we can assume $s = 0$. Conjugating by b, we can also assume $r = 0$. Now the product can be grouped into terms which are either ab or ab^2, mapping to $S(ST) = x$ or $S(ST)^2 = TST = y$. From the lemma, this cannot be zero unless the original product was empty. Hence the map is injective.

(I took this cute proof from Newman, "Integer Matrices", Academic Press 1972, §VIII.3. A more powerful approach comes from the general theory of Fuchsian groups, which allows one to deduce a presentation for G directly from the shape of the fundamental domain $\mathcal{D}$; for this argument see Lehner, "Discontinuous groups and automorphic functions", AMS 1963, §VII.2.)