MA4H9 Modular Forms: Problem Sheet 3 – Solutions

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This is the third of three problem sheets, each of which amounts to 5% of your final mark for the course. This problem sheet will be marked out of a total of 40; the number of marks available for each question is indicated. You should hand in your work to the Undergraduate Office by ***3pm on Monday 10th January 2011****.*

Throughout this sheet, Γ is a finite-index subgroup of $SL_2(\mathbb{Z})$ and $k \in \mathbb{Z}$ (some questions will make additional assumptions on k).

1. [6 points] Let f be a nonzero modular function of level Γ and weight k. Recall that the total valence $V_\Gamma(f)$ was defined by

$$V_\Gamma(f) = \sum_{c \in C(\Gamma)} v_{\Gamma,c}(f) + \sum_{z \in \Gamma \backslash \mathcal{H}} \frac{v_z(f)}{n_\Gamma(z)}.$$

Give a careful proof of Lemma 2.4.5, which states that for $g \in SL_2(\mathbb{Z})$ the total valence of f and $f\mid_k g$ are related by

$$V_{g^{-1}\Gamma g}(f \mid_k g) = V_\Gamma(f).$$

Solution: In lectures, we saw that in order to prove 2.4.5 it suffces to show that

(a) $n_\Gamma(z) = n_{g^{-1}\Gamma g}(g^{-1}z)$

(b) $v_z(f|_k g) = v_{gz}(f)$ for all $z \in \mathcal{H}$

(c) $v_{g^{-1}\Gamma g,c}(f|_k g) = v_{\Gamma,gc}(f)$ for all $c \in C(g^{-1}\Gamma g)$.

For part (a), the map sending $\overline{x} \in \overline{\Gamma}$ to $\overline{g^{-1}xg}$ gives a bijection between the sets $\mathrm{Stab}_{\overline{\Gamma}}(z)$ and $\mathrm{Stab}_{\overline{g^{-1}\Gamma g}}(g^{-1}z)$, so the two sets have the same size.

Recall that $(f|_k g)(z) := j(g,z)^{-k} f(gz)$. Since $j(g,z)$ is holomorphic and non-vanishing on $\mathcal{H}$, it's enough to check that the order of vanishing of $w \mapsto f(gw)$ at $w = z$ is the same as the order of vanishing of $w \mapsto f(w)$ at $w = gz$. This is not quite automatic; it's true because for w sufficiently close to z, we have $g(w) = g(z) + (w-z)g'(z) + O((w-z)^2)$, and (crucially) $g'(z)$ is never zero on $\mathcal{H}$. The result now follows by substituting this into the Taylor series of f around gz.

Finally, part (c) is essentially automatic from the definition of $v_{gc}(f)$: if h is some element mapping ∞ to c, then gh maps ∞ to gc, and both $v_{g^{-1}\Gamma g,c}(f|_k g)$ and $v_{\Gamma,gc}(f)$ are (by definition) equal to to $v_{h^{-1}g^{-1}\Gamma gh,\infty}(f|_k gh)$.

[Several of you slipped up at part (b); there were several solutions which relied on the "identity" $gz - gz_0 = g(z - z_0)$ for $z, z_0 \in \mathcal{H}$. This is false, and might not even be meaningful since $z - z_0$ isn't necessarily in $\mathcal{H}$; the map $\mathcal{H} \to \mathcal{H}$ given by the action of g isn't linear, but because g has nonzero derivative everywhere, we can approximate it by a linear map in the neighbourhood of each point.]

2. [3 points] In my first research paper, I found myself needing the following identity of weight 0 modular functions of level $\Gamma_0(2)$:

$$\frac{E_6^2}{\Delta} = \frac{(1+2^6 f_2)(1-2^9 f_2)^2}{f_2}$$

where $f_2(z) = \frac{\Delta(2z)}{\Delta(z)}$. I verified by a computer calculation that the q-expansions of both sides agreed up to q^N, for some sufficiently large N. How large an N did I need to use? *(Hint: Clear denominators and apply Corollary 2.4.7).*

Solution: Let A and B stand for the left and right sides of the formula above. If we abbreviate $\Delta(2z)$ by Δ_2, and multiply both sides by $\Delta^2\Delta_2$, we note that

$$\Delta^2\Delta_2 A = \Delta\Delta_2 E_6^2,$$
$$\Delta^2\Delta_2 B = (\Delta + 2^6\Delta_2)(\Delta - 2^9\Delta_2)^2,$$

both of which are clearly in $M_{36}(\Gamma_0(2))$. Hence if they agree up to degree

$$\frac{36 \times d_{\Gamma_0(2)}}{12} = 9,$$

they are equal, by corollary 2.4.7 (the unreasonable effectiveness of modular forms). This is certainly the case if A and B agree up to degree q^5, since the leading term of the q-expansion of $\Delta^2\Delta_2$ is q^4.

[Slightly better bounds are possible by keeping track of the orders of vanishing of both sides at 0 *and* ∞ *simultaneously; but any valid argument giving a finite bound got full marks.*

I was a bit concerned that at least three of you thought that $\Delta^2\Delta_2$*, as the product of three weight 12 forms, had weight* 12^3*, which led to some spuriously large bounds (*$N = 288$ *came up more than once).*

Also, several people rewrote the claim as $E_6^2 f_2 = (1 + 2^6 f_2)(1 - 2^9 f_2)^2\Delta$ *and applied 2.4.7 in weight 12, disregarding the fact that* f_2 *is not a modular form – it has a pole at the cusp* 0*.]*

3. [2 points] Show that commensurability is an equivalence relation on the set of subgroups of a fixed group G.

Solution: We must check that the relation $\sim$ of commensurability is reflexive ($A \sim A$ for all A), symmetric ($A \sim B \Leftrightarrow B \sim A$), and transitive (if $A \sim B$ and $B \sim C$, then $A \sim C$). The first two are self-evident, so let us suppose that A, B, C are subgroups satisfying $A \sim B$ and $B \sim C$. Then $A \cap B$ has finite index in A, and $B \cap C$ has finite index in B.

We need the following easy lemma: if D is any group, and E and F are any subgroups of D with E finite-index, then $E \cap F$ has finite index in F and $[F : E \cap F] \leq [D : E]$. (We have used this several times in the course already.) This follows from the fact that there is a natural bijection $F/E \cap F \leftrightarrow EF/E$, and $EF/E \subseteq D/E$.

We apply this result with $D = B$, $E = B \cap C$, and $F = A$. This tells us that $A \cap B \cap C$ has finite index in $A \cap B$. Since $A \cap B$ in turn has finite index in A, it follows that $A \cap B \cap C$ has finite index in A; hence $A \cap C$ has finite index in A. Arguing similarly, $A \cap C$ has finite index in C as well; this proves that $A \sim C$.

[Everybody got full marks for this question – good work.]

4. [3 points] Let $k \geq 1$ and let $N_k(\Gamma)$ be the Eisenstein subspace of $M_k(\Gamma)$ (defined as the orthogonal complement of $S_k(\Gamma)$ with respect to the Petersson product). Show that $N_k(\Gamma)$ is preserved by the action of $[\Gamma g \Gamma]$ for any $g \in \mathrm{GL}_2^+(\mathbb{Q})$.

Solution: Recall that on a positive-definite inner product space V, an operator A preserves a subspace $W \subseteq V$ if and only if the adjoint A^* preserves the complementary subspace $V^\perp$. Since the adjoint of $[\Gamma g \Gamma]$ is given by $[\Gamma g' \Gamma]$, where $g' = (\det g) g^{-1}$ is also in $\mathrm{GL}_2^+(\mathbb{Q})$, it suffices to note that $S_k(\Gamma)$ is preserved by $[\Gamma g \Gamma]$ for all $g \in \mathrm{GL}_2^+(\mathbb{Q})$.

[This question was also done well by most of you.]

5. [3 points] Let f be the unique normalised eigenform in $S_2(\Gamma_0(11))$, and let $g = f^2$. Calculate the first two terms of the q-expansions of g and of $T_2(g)$, and hence show that $\dim S_4(\Gamma_0(11)) \geq 2$.

Solution: We saw in lectures that the q-expansion of f is given by $q\prod_{n\geq 1}(1-q^n)^2(1-q^{11n})^2 = q - 2q^2 - q^3 + O(q^4)$. Squaring this term-by-term, we deduce that $g = q^2 - 4q^3 + 2q^4 + O(q^5)$. (Note that we will need terms of g up to q^4 in order to calculate $T_2(g)$ up to q^2).

Applying T_2 using the usual q-expansion formulae, we get $T_2(g) = q + 2q^2 + O(q^3)$; this is clearly not a multiple of g, so the space has dimension ≥ 2.

[Almost all of you did this fine, some by hand and others using Sage. It is perhaps cheating slightly if you let Sage calculate the q-expansion of f for you, rather than using the η-product formula from the notes, but never mind.]

6. [6 points] Let $N \geq 2$ and let χ be a Dirichlet character mod N. Let $a \in \mathbb{Z}/N\mathbb{Z}$ and define the *Gauss sum*

$$\tau(a, \chi) = \sum_{b \in (\mathbb{Z}/N\mathbb{Z})^\times} e^{2\pi i ab/N} \chi(b).$$

(a) Show that $\tau(a, \chi) = \overline{\chi(a)}\tau(1, \chi)$ if $a \in (\mathbb{Z}/N\mathbb{Z})^\times$.

Solution: Since $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, multiplication by a gives a bijection from $(\mathbb{Z}/N\mathbb{Z})^\times$ to itself. Thus we may substitute a new variable $c = ab$ in the sum defining $\tau(a, \chi)$ to obtain

$$\begin{aligned}\tau(a, \chi) &= \sum_{c \in (\mathbb{Z}/N\mathbb{Z})^\times} e^{2\pi i c/N} \chi(a^{-1}c) \\ &= \chi(a^{-1}) \sum_{c \in (\mathbb{Z}/N\mathbb{Z})^\times} e^{2\pi i c/N} \chi(c) \\ &= \overline{\chi(a)}\tau(1, \chi).\end{aligned}$$

(We have $\chi(a^{-1}) = \chi(a)^{-1}$ since χ is a homomorphism, and $\chi(a)^{-1} = \overline{\chi(a)}$ since $\chi(a)$ is a root of unity.)

(b) Let $M \mid N$. Show that if χ does not factor through $(\mathbb{Z}/M\mathbb{Z})^\times$, then we have

$$\sum_{\substack{b \in (\mathbb{Z}/N\mathbb{Z})^\times \\ b = 1 \bmod M}} \chi(b) = 0.$$

Solution: If χ does not factor through $(\mathbb{Z}/M\mathbb{Z})^\times$, then there is some $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ with $a = 1 \bmod M$ such that $\chi(a) \neq 1$. Multiplication by a is a bijection on the set $\{b \in (\mathbb{Z}/N\mathbb{Z})^\times : b = 1 \bmod M\}$, so we have

$$\sum_{\substack{b \in (\mathbb{Z}/N\mathbb{Z})^\times \\ b = 1 \bmod M}} \chi(b) = \sum_{\substack{b \in (\mathbb{Z}/N\mathbb{Z})^\times \\ b = 1 \bmod M}} \chi(ab) = \chi(a) \sum_{\substack{b \in (\mathbb{Z}/N\mathbb{Z})^\times \\ b = 1 \bmod M}} \chi(b).$$

Since $\chi(a) \neq 1$, this forces the sum to be 0.

Hence show that if χ is primitive, $\tau(a, \chi) = 0$ for $a \neq (\mathbb{Z}/N\mathbb{Z})^\times$.

Solution: Suppose $a \neq (\mathbb{Z}/N\mathbb{Z})^\times$. Then there is some $M \mid N$, $M < N$, such that a is a multiple of N/M. Thus, for $b, c \in (\mathbb{Z}/N\mathbb{Z})^\times$, we have $e^{2\pi i a b/N} = e^{2\pi i a c/N}$ if $b = c \pmod{M}$. Thus

$$\tau(a,\chi) = \sum_{b \in (\mathbb{Z}/N\mathbb{Z})^\times} e^{2\pi i a b/N} \chi(b) = \sum_{c \in (\mathbb{Z}/M\mathbb{Z})^\times} e^{2\pi i a c/N} \left(\sum_{\substack{b \in (\mathbb{Z}/M\mathbb{Z})^\times \\ b = c \bmod M}} \chi(b) \right).$$

The bracketed term is a multiple of $\sum_{\substack{b \in (\mathbb{Z}/M\mathbb{Z})^\times \\ b = 1 \bmod M}} \chi(b)$, which is zero.

(c) Calculate $\tau(1,\chi)$ when $N = p^j$ (p prime, $j \geq 1$) and χ is the trivial character mod p^j.

Solution: By definition, we have

$$\tau(1,\chi) = \sum_{\substack{b = 0 \dots p-1 \\ p \nmid b}} e^{2\pi i b/p^j}.$$

This is the sum of the primitive p^jth roots of 1; that is, it is the sum of the roots of $X^{p^j} - 1$ which are not also roots of $X^{p^{j-1}} - 1$, or the roots of the *cyclotomic polynomial*

$$\Phi_{p^j}(X) = \frac{X^{p^j} - 1}{X^{p^{j-1}-1}} = 1 + X^{p^{j-1}} + \cdots + X^{(p-1)p^{j-1}}.$$

Since the sum of the roots of a monic polynomial of degree d is -1 times its coefficient of X^{d-1}, this implies that the sum is -1 if $j = 1$ and 0 otherwise.

7. [4 points] Let $N \geq 2$ and let H be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. Define $\widehat{H}$ to be the subgroup of Dirichlet characters χ mod N such that $\chi(d) = 1$ for all $d \in H$.

(a) Show that $\Gamma_H(N) = \left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N) : a, d \in H \right\}$ is a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Solution: It's clear that $a \in H$ if and only if $d \in H$, since $ad = ad - bcN = 1 \bmod N$; and the map $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ sending $\begin{pmatrix} a & b \\ cN & d \end{pmatrix}$ to d is a surjective homomorphism with kernel $\Gamma_1(N)$, by a question from sheet 2. Hence $\Gamma_H(N)$ is the preimage of a subgroup under a homomorphism; so it is a subgroup of $\Gamma_0(N)$. It clearly contains $\Gamma_1(N)$, so it has finite index in $\mathrm{SL}_2(\mathbb{Z})$.

[This was a very easy sub-question; everyone who attempted it got the available 2 marks. I'm puzzled that four of you didn't even try it.]

(b) Show that for any $k \geq 1$ we have

$$S_k(\Gamma_H(N)) = \bigoplus_{\chi \in \widehat{H}} S_k(\Gamma_1(N), \chi).$$

Solution: If $f \in S_k(\Gamma_1(N), \chi)$, and $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_H(N)$, then $f|_k\gamma = \langle d \rangle f = \chi(d) f$. So $f|_k\gamma = f$ for all $\gamma \in \Gamma_H(N)$ if and only if $\chi(d) = 1$ for all $d \in H$.

It follows that $S_k(\Gamma_1(N),\chi) \subseteq S_k(\Gamma_H(N))$ if $\chi \in \widehat{H}$ and $S_k(\Gamma_1(N),\chi) \cap S_k(\Gamma_H(N)) = 0$ otherwise. Since $S_k(\Gamma_1(N)) = \bigoplus_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} S_k(\Gamma_1(N),\chi)$, the result follows.

[I think I managed to confuse several of you by a notational inconsistency. In the lectures, I defined $\widehat{G}$ for an arbitrary abelian group G to be the group of characters $G \to \mathbb{C}^\times$. Here $\widehat{H}$ isn't the characters of H, but the characters of G trivial on H, or (equivalently) the characters of the quotient G/H. I should perhaps have called this something different, perhaps $H^\perp$ or $H^\vee$. The misleading notation fooled you into thinking that this was an instance of proposition 2.9.3, which it isn't quite.]

8. [4 points] Suppose p is a prime, $\Gamma = \Gamma_1(p)$ and $g = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Find p matrices $(g_j)_{j=0,\dots,p-1}$ in $\mathrm{GL}_2^+(\mathbb{Q})$ such that

$$\Gamma g \Gamma = \bigsqcup_{0 \le j < p} \Gamma g_j = \bigsqcup_{0 \le j < p} g_j \Gamma.$$

Solution: We first find a_j such that $\Gamma g \Gamma = \bigsqcup \Gamma a_j$. It suffices to take $a_j = g a_j'$, where a_j are a set of coset representatives for the left coset space $(\Gamma \cap g^{-1}\Gamma g)\backslash\Gamma$. We find that $\Gamma \cap g^{-1}\Gamma g$ is the group $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : b = 0 \bmod p\}$, and a set of coset representatives is given by $a_j' = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$ for $j = 0, \dots, p-1$. We have $a_j = g a_j' = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$, so we recover the result stated in lectures that

$$\Gamma g \Gamma = \bigsqcup_{j=0}^{p-1} \Gamma \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}.$$

We now do the "opposite" decomposition; that is, we find b_j such that $\Gamma g \Gamma = \bigsqcup \Gamma b_j$. Now we take $b_j = b_j' g$ where b_j' are coset representatives for the right coset space $\Gamma/(\Gamma \cap g\Gamma g^{-1})$. We find that $\Gamma \cap g\Gamma g^{-1}$ is the group $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c = 0 \bmod p^2\}$, and a set of coset reps is given by $b_j' = \begin{pmatrix} 1 & 0 \\ pj & 1 \end{pmatrix}$. We have $b_j = b_j' p = \begin{pmatrix} 1 & 0 \\ pj & p \end{pmatrix}$, so we have

$$\Gamma g \Gamma = \bigsqcup_{j=0}^{p-1} \begin{pmatrix} 1 & 0 \\ pj & p \end{pmatrix} \Gamma.$$

Let's find the intersection of the cosets $\Gamma \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ pj & p \end{pmatrix} \Gamma$. Let $\gamma = \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in \Gamma$. We find that

$$\begin{aligned} & \gamma \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ pj & p \end{pmatrix} \Gamma \\ \iff & \begin{pmatrix} 1 & 0 \\ -j & \frac{1}{p} \end{pmatrix} \gamma \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \in \Gamma \\ \iff & \begin{pmatrix} a & aj + bp \\ -aj + c & -aj^2 + cj - bjp + d \end{pmatrix} \in \Gamma. \end{aligned}$$

Looking at the left column suggests trying $a = 1$ and $j = c$, in which case the requirement that $\det \gamma = 1$ forces $d = 1 + bjp$. We may as well try $b = 0$, and then the above messy matrix works out as $\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$, which is certainly in Γ. Hence we conclude that

$$\begin{pmatrix} 1 & 0 \\ pj & 1 \end{pmatrix} \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & j \\ pj & p + pj^2 \end{pmatrix} \in \Gamma \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \cap \begin{pmatrix} 1 & 0 \\ pj & p \end{pmatrix} \Gamma.$$

Thus taking $g_j = \begin{pmatrix} 1 & j \\ pj & p + pj^2 \end{pmatrix}$ works.

[Sadly, absolutely everyone who tried this question tried to show that one could take $g_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$. *This does not work, meaning that nobody got more than 1 mark. We do have*

$$\Gamma g \Gamma = \bigsqcup_j \Gamma \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix},$$

*but **all** of the matrices* $\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$ *lie in the single right coset* $g\Gamma$ *– not surprisingly, since* $g_j = g \cdot \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \in \Gamma$ *– so they certainly don't work the other way round. It's important to distinguish between the roles of the two subgroups* $\Gamma \cap g^{-1}\Gamma g$ *and* $\Gamma \cap g\Gamma g^{-1}$. *]*

9. [5 points] Let V be a finite-dimensional complex vector space endowed with a positive definite inner product (a finite-dimensional Hilbert space). Let $A : V \to V$ be a linear operator.

   (a) Show that if A is selfadjoint, $\langle Ax, x \rangle$ is real for all $x \in V$.

   Solution: We have $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$, by the definition of an inner product. On the other hand, since A is selfadjoint we have $\langle Ax, x \rangle = \langle x, Ax \rangle$. Thus $\langle Ax, x \rangle = \overline{\langle Ax, x \rangle}$, so $\langle Ax, x \rangle$ is real.
   [All of you got this.]

   (b) We say A is *positive semidefinite* if it is selfadjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in V$. Show that if A is positive semidefinite, there is a unique positive semidefinite B such that $B^2 = A$. (We write $B = \sqrt{A}$.)

   Solution: Since A is selfadjoint, it is certainly normal, and thus diagonalisable. Hence we may choose a basis such that A is diagonal. Moreover, the eigenvalues of A must be nonnegative real numbers, since if v is an eigenvector with eigenvalue λ we have $\langle Av, v \rangle = \lambda \langle v, v \rangle$. Since a nonnegative real number has a nonnegative real square root, we can set B to be the diagonal matrix whose entries are the nonnegative real square roots of those of A; then B is clearly positive semidefinite with $B^2 = A$.
   It remains to show uniqueness. Since A is diagonalisable and B must commute with A and hence preserves its eigenspaces, it suffices to check that a matrix of the form λI, where $\lambda \geq 0$ and I is the identity matrix, has a unique positive definite square root. But any candidate square root must be diagonalisable with all diagonal entries equal to $\sqrt{\lambda}$; so it is conjugate to a scalar multiple of the identity, and thus it is the identity.
   [Catastrophically, almost everyone got this wrong; since there is only one mark available, I couldn't give that mark to anyone who didn't have a complete proof of both the existence and uniqueness. Most of you proved existence, but only a few even remembered to check uniqueness, and all but one of those who did so gave arguments that are only valid if the eigenspaces of A are all one-dimensional.]

   (c) Show that for any linear operator A, the operator A^*A is positive semidefinite, and if $P = \sqrt{A^*A}$, then we may write $A = UP$ with U unitary. Show conversely that if $A = UP$ with U unitary and P positive semidefinite, we must have $P = \sqrt{A^*A}$. Is U uniquely determined?

   Solution: We have $(A^*A)^* = A^*(A^*)^* = A^*A$, so A^*A is selfadjoint; and $\langle A^*Ax, x \rangle = \langle Ax, Ax \rangle \geq 0$ since the inner product is positive definite.

Let us write $P = \sqrt{A^*A}$. If A is non-singular, then P is so also (since $\det P = |\det A|$). Hence we may define $U = AP^{-1}$. We find that

$$P^2 = A^*A = PU^*UP.$$

Since we are assuming that A is nonsingular, we can cancel P from both sides and deduce that $U^*U = 1$, i.e. U is unitary; and it is clear that U is the only matrix (unitary or otherwise) such that $A = UP$.

If A is singular, the argument is a little more delicate. Let $V_1 = \ker(P)$ and $V_2 = \mathrm{im}(P)$. Because P is selfadjoint, we find that $V_1 = V_2^\perp$ and vice versa. One checks that A is zero on V_1, and $AV_2 \subseteq V_2$. Thus we can define U by taking the direct sum of an arbitrary unitary operator $V_1 \to V_1$, and the uniquely defined unitary part of A restricted to V_2.

Conversely, if $A = UP$ with U unitary and P positive semidefinite, we must have $A^*A = PU^{-1}UP = P^2$, so necessarily $P = \sqrt{A^*A}$; but from the above argument it is clear that U is uniquely determined if and only if A is invertible.

[A few of you gave correct arguments under the assumption that A was nonsingular, which I gave the mark to. Only two gave complete arguments for the general case.]

(d) Show that A is normal if and only if we can find a unitary U and positive semidefinite P such that $A = UP$ and U and P commute.

Solution: In fact more is true: if A is normal, then for *any* such factorisation $A = UP$, we must have $UP = PU$. Indeed, we have $A^*A = AA^*$ (since A is normal) and $AA^* = (UP)(UP)^* = UP^2U^{-1} = (UPU^{-1})^2$. One checks easily that UPU^{-1} is positive semidefinite. So P and UPU^{-1} are both positive semidefinite self-adjoint operators squaring to A^*A; by the uniqueness from part (b), this implies that $P = UPU^{-1}$, so U and P commute.

(Alternatively, one can construct a commuting U and P directly, by using the fact that A is diagonalisable; but this does not give the slightly stronger statement above.)

Conversely, if there exists some decomposition $A = UP$ with U and P commuting, then $A^*A = P^2$ and $AA^* = UP^2U^{-1} = P^2$; so $AA^* = A^*A$, i.e. A is normal.

[A common mistake here was to observe that $AA^* = A^*A$ is equivalent to $UP^2 = P^2U$ and to claim that this implies immediately that $UP = PU$. It's not true for general linear operators that if A and B^2 commute, A and B necessarily commute – for a counterexample, try $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and A any non-diagonal matrix.]

(e) Find an example of a nondegenerate (but not positive definite!) inner product space V and a linear operator $A : V \to V$ which is normal but not diagonalisable.

Solution: The simplest example I can think of is to take $V = \mathbb{C}^2$, with the inner product defined by $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 0$ and $\langle e_1, e_2 \rangle = 1$. Then the adjoint of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}$. In particular the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is selfadjoint, and hence normal, but it is not diagonalisable.

[Three of you found this example, and one of you found a 3-dimensional example, which actually turns out to have this one secretely living inside it.]

10. [4 points] Let f be weakly modular of level Γ and weight k. Let $f^* : \mathcal{H} \to \mathbb{C}$ be the function defined by $f^*(z) = \overline{f(-\bar{z})}$. Show that f^* is weakly modular of weight k and level $\Gamma^* = \sigma^{-1}\Gamma\sigma$, where $\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution: *[You all seem to have found this rather hard. I had one complete solution that deduced the result from some rather advanced considerations involving Galois theory, but there is a simple direct proof that nobody gave:]*

By assumption, we have

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Hence

$$f\left(\frac{a(-\overline{z})+b}{c(-\overline{z})+d}\right) = (c(-\overline{z})+d)^k f(-\overline{z})$$

or (rearranging the fraction on the left-hand side)

$$f\left(-\overline{\left(\frac{az-b}{-cz+d}\right)}\right) = (c(-\overline{z})+d)^k f(-\overline{z}).$$

Conjugating everything in sight, we've shown

$$f^*\left(\frac{az-b}{-cz+d}\right) = (-cz+d)^k f^*(z).$$

That is, f^* is modular of weight k for the group $\left\{\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right\}$, which is precisely Γ^*.

A slightly slicker interpretation of this is: let X be the space of meromorphic functions on $\mathbb{C}\backslash\mathbb{R}$ which satisfy $f(z) = \overline{f(\overline{z})}$. There's a unique way to extend any given meromorphic function on $\mathcal{H}$ to an element of X. Then it's easy to see that the weight k action of $\mathrm{GL}_2^+(\mathbb{R})$ extends to an action of the whole of $\mathrm{GL}_2(\mathbb{R})$ on X, and f^* is just $f|_k\sigma$.

Show that f^* is a modular function, modular form, or cusp form if and only if f is, and that the q-expansions of f^* and f at ∞ are related by $a_n(f^*) = \overline{a_n(f)}$.

Solution: Let us first consider behaviour at ∞. It's clear that $(\Gamma^*)_\infty = \Gamma_\infty$, so $h_\Gamma(\infty) = h_{\Gamma^*}(\infty)$. Moreover, for any integer (or real) h we have

$$\begin{aligned}\overline{q_h(-\overline{z})} &= \overline{\exp(2\pi i(-\overline{z})/h)} \\ &= \exp(\overline{2\pi i(-\overline{z})/h}) \\ &= \exp(2\pi\overline{(i)}\;\overline{(-\overline{z})}/h) \\ &= \exp(2\pi(-i)(-z)/h) = q_h(z).\end{aligned}$$

It follows that if $f(z)$ is given by a series $\sum_{n=-N}^{\infty} a_n q_h(nz)$ for some $N < \infty$, converging for all $\mathrm{Im}(z)$ sufficiently large, then (in the same range of $\mathrm{Im}(z)$) we have $f^*(z) = \sum \overline{a_n} q_h(nz)$, and conversely. So f^* is meromorphic at ∞ if and only if f is, and if so we have $v_{\Gamma^*,\infty}(f^*) = v_{\Gamma,\infty}(f)$ and $a_n(f^*) = \overline{a_n(f)}$. (We have assumed here that Γ is regular at infinity or that k is even; the argument goes through almost identically in the odd weight irregular case.)

For a general cusp $c \in \mathbb{P}^1(\mathbb{Q})$, let $g \in \mathrm{SL}_2(\mathbb{Z})$ be such that $g\infty = c$. Then $(\sigma^{-1}g\sigma)(\infty) = -c$. Hence, by definition, we have

$$\begin{aligned} v_{\Gamma,c}(f) &= v_{g^{-1}\Gamma g,\infty}(f|_k g) \\ v_{\sigma^{-1}\Gamma\sigma,-c}(f^*) &= v_{(\sigma^{-1}g^{-1}\sigma)(\sigma^{-1}\Gamma\sigma)(\sigma^{-1}g\sigma),\infty}(f^*|_k\sigma^{-1}g\sigma).\end{aligned}$$

The rather messy expression $(\sigma^{-1}g^{-1}\sigma)(\sigma^{-1}\Gamma\sigma)(\sigma^{-1}g\sigma)$ simplifies down to just $\sigma^{-1}g^{-1}\Gamma g\sigma = (g^{-1}\Gamma g)^*$. Moreover, it's easy to check that $(f^*)|_k(\sigma^{-1}g\sigma) = (f|_k g)^*$ (this is the same argument as we used above to prove the first sentence of the question). Hence applying the preceding argument with f and Γ replaced by $f|_k g$ and $g^{-1}\Gamma g$, we deduce that f^* is meromorphic at $-c$ if and only if f is merormorphic at c, and if so then we have $v_{\Gamma,c}(f) = v_{\Gamma^*,-c}(f^*)$.

In particular, we see that f^* is meromorphic / holomorphic / vanishing at all cusps if and only if f is; and since f^* is obviously meromorphic or holomorphic on $\mathcal{H}$ if and only if f is, we deduce that f^* is a modular function (respectively modular form or cusp form) if and only if this is true of f.

11. *[Non-assessed and for amusement only]* Let $M(\Gamma) = \bigoplus_{k\geq 0} M_k(\Gamma)$, which is clearly a ring. Show that for any Γ, $M(\Gamma)$ is finitely generated as an algebra over $\mathbb{C}$, and we may take the generators to have weight at most 12.

Solution: Nobody did this question, sadly. Nils Skoruppa told me this at a conference once; I don't know why it's true.