

Problem sheet #2 on
 course web page.
 Deadline in 2 wks.

Recall: for S scheme, $\alpha, \beta \in \Gamma(S, \mathcal{O}_S)$,

$E(\alpha, \beta) = \text{curve def by}$
 $Y^2 + \alpha XY + \beta Y = X^3 + \beta X^2$
 defines an elliptic curve if $\Delta(\alpha, \beta) \in \Gamma(S, \mathcal{O}_S)^\times$.

Prop 3.3.4 For any S , E/S elliptic curve,
 $P \in E(S)$ st $P, 2P, 3P \neq 0$ on every
 fibre, $\exists! \alpha, \beta \in \Gamma(S, \mathcal{O}_S)$ and
 iso $E \cong E(\alpha, \beta)$ st $P \mapsto (0, 0)$.

Proof First, assume E has a Weierstrass
 eq^n over S .

By a translation $x \mapsto x+t, y \mapsto y+u$
 can assume $P = (0, 0)$.
 Since P does not have order 2 in any fibres,
 gradients of tangent line at P is $u(S) \neq 0$
 by replacing y with $y+ux$ some u can put
 eq^n in form

$$Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2.$$

$a_i \in \Gamma(S, \mathcal{O}_S)$.

Since P does not have order 3 in any
 fibres, $(0, 0)$ not an inflexion pt, so
 $a_2 \in \Gamma(S, \mathcal{O}_S)^\times$

So by α scaling $x \mapsto u^2 x, y \mapsto u^3 y$
 can arrange that $a_2 = a_3$. Then E is
 $E(\alpha, \beta)$ where $\alpha = a_1, \beta = a_2$.

This gives an isomorphism to a curve in
 Tate normal form.

Now consider a general E/S . We know

\exists covering $S = \bigcup U_i$ st $E|_{U_i}$
 has a Weierstrass eq^n over $\Gamma(U_i, \mathcal{O}_S)$.

So get $\alpha_i, \beta_i \in \Gamma(U_i, \mathcal{O}_S)$ st
 $(E|_{U_i}, P|_{U_i}) \cong (E(\alpha_i, \beta_i), (0, 0))$

Since α_i, β_i are unique, must agree on

$U_i \cap U_j$: Sheaf property of $\mathcal{O}_S \Rightarrow$

$\exists \alpha, \beta \in \Gamma(S, \mathcal{O}_S)$ st $\text{res}_{U_i}(\alpha) = \alpha_i$
 $\text{res}_{U_i}(\beta) = \beta_i \forall i$

Then $(E, P) \cong (E(\alpha, \beta), (0, 0))$

Remark Last step used in an essential
 way, the uniqueness of (α, β) :
 "local uniqueness gives global existence"

Corollary 3.3.5 (i)

The pair
 $(\text{Spec } \mathbb{Z}[A, B, \Delta(A, B)^{-1}], (E(A, B), (0, 0)))$

represents the functor

$$S \mapsto \left\{ \begin{array}{l} \text{equiv. classes } (E, P), \\ E/S \text{ ell. curve, } P \text{ pt. of order } \\ 1, 2, 3 \text{ in any fiber} \end{array} \right\}$$

(ii) The pair
 $(\text{Spec } \mathbb{Z}[B, \Delta(4B, B)^{-1}], (E(4B, B), (0, 0)))$

represents

$$S \mapsto \left\{ (E, P) \mid \begin{array}{l} E/S \text{ ell. curve +} \\ P \text{ pt. of exact order } \\ 5 \text{ (in every fib)} \end{array} \right\}$$

Proof (i) is a restatement of Prop 3.3.4

(ii): just equate $3P = -2P$

$$\begin{array}{l} 3P = (1-A, A-B-1) \\ -2P = (-B, 0) \end{array} \quad \square$$

Note $\Delta(4B, B) = B^5 \underbrace{(B^2 + 4B - 1)}_{\text{discriminant } S^3}$

Is it reasonable to define

$$Y_1(S)_{\mathbb{Z}} \text{ to be } \text{Spec } \mathbb{Z}[B, \Delta(4B, B)^{-1}]$$

Sadly no: our defⁿ of "pt. of exact order 5"
is too naive in char. 5.

If E/\mathbb{F}_5 is supersingular, $E[S]$

is a single point with multiplicity 25.

- no points of order 5, even over \mathbb{F}_5 .

So this scheme $\text{Spec } \mathbb{Z}[-]$ has empty fib/ supersingular j -invs.

Defⁿ 3.3.6 We set

$$Y_1(S)_{\mathbb{Z}[\frac{1}{5}]} = \text{Spec } \mathbb{Z}[\frac{1}{5}, B, \Delta(4B, B)^{-1}]$$

+ this represents the same functor as before
on cat. of $\mathbb{Z}[\frac{1}{5}]$ -schemes.

More generally:

Def 3.3.6 ($\ell \neq 4$):

for $N \geq 4$, let $Y_N =$ closed subscheme of

$$Y = \text{Spec } \mathbb{Z}[A, B, \Delta(A, B)^{-1}] \text{ where } N \cdot (0, 0, 1)$$

and let $(0, 1, 0)$

$$Y_1(N)_{\mathbb{Z}[\frac{1}{N}]} = \left(\bigcup_{N \geq d} Y_d - \bigcup_{\substack{d|N \\ 4 \leq d < N}} Y_d \right) \times_{\text{Spec } \mathbb{Z}[\frac{1}{N}]} \text{Spec } \mathbb{Z}[\frac{1}{N}]$$

By construction this represents

$$S \mapsto \left\{ \text{ell. curves } E/S \text{ w. pt. of exact} \right.$$

order N
on category of $\mathbb{Z}[\frac{1}{N}]$ -schemes.

(More precisely,
 $Y_1(N)_{2\mathbb{Z}(\frac{1}{N})}$ has a universal elliptic curve
over it by restricting $E(N, \mathbb{B}) / Y$,
 \dagger this has a pt $(0, 0)$.
and $(Y_1(N)_{2\mathbb{Z}(\frac{1}{N})})$, (this one, this pt.)
represents the above functor.)

Two natural questions

- (1) What does $Y_1(N)_{2\mathbb{Z}(\frac{1}{N})}$ look like?
Is it non-singular?
- (2) \exists bijection of sets between
 $Y_1(N)_{2\mathbb{Z}(\frac{1}{N})}(\mathbb{C})$ and $\mathbb{P}^1(\mathbb{C})$.
Is this a map of alg. varieties / \mathbb{C} ?

§ 3.4 Smoothness

Def 3.4.1 A morphism of schemes
 $\phi: X \rightarrow Y$ is smooth if it's
locally of finite presentation, flat,
and for every point $y \in Y$, the fibre $\phi^{-1}(y)$
is a smooth variety over $k(y)$.
(So our def's of elliptic curves / S require that
 $E \rightarrow S$ be a smooth morphism.)

Lemma 3.4.2

- (i) The composition of smooth morphisms
is smooth.
- (ii) If E/S is an elliptic curve and $N > 1$ is
invertible on S , then $[N]: E \rightarrow E$ is
smooth.

Proof (i) is standard (see EGA)
(follow trail of references
from Wikipedia)

- (ii) The morphism $[N]$ multiplies a
global differential by N , so induces an
isomorphism on tangent space,
i.e. it's an étale morphism. (and étale morphisms
are smooth.) \square

Prop 3.4.3 (Functional criterion for smoothness)

Let $X \rightarrow \text{Spec } R$ be a scheme of
finite type / R , R Noetherian.

The map $X \rightarrow \text{Spec } R$ is a smooth morphism
if and only if it's "formally smooth", i.e. for
any $\mathbb{Z}[R]$ -algebra A and nilpotent ideal $I \subset A$,
the map

$$\text{Hom}_{\text{Sch}/R}(\text{Spec } A, X) \rightarrow \text{Hom}_{\text{Sch}/R}(\text{Spec } A_0, X)$$

is surjective, where $A_0 = A/I$.

(If we replace "surj" with "bijective" get notion of
"formally étale".)
Proof See Stacks Project §3C.9. \square

Thm 3.44

$\mathcal{Y}_i(N)_{\mathbb{Z}[\frac{1}{N}]}$ is smooth over $\mathbb{Z}[\frac{1}{N}]$.

Proof Let A be a local $\mathbb{Z}[\frac{1}{N}]$ -alg,
 $I \subset A$ nilpotent.

Let $(E, P) \in \mathcal{Y}_i(N)(A_0)$.

A_0 is local, so E_0 has a Weierstrass eqⁿ
over $\text{Spec}(A_0)$.

Lift coeffs arbitrarily to A to get an
 E/A lifting E_0 (discriminant $\Delta(E)$ is in A^\times
as its image in A_0 is in A_0^\times .)

Can we lift P_0 to an N -torsion pt on E ?

In other words, is $E[N]$ smooth? But it is,
because $[N]: E \rightarrow E$ is smooth and
a composition of smooth morphisms is smooth
($[N]$ composed with structure map $E \rightarrow \text{Spec} A$).

So (E, P) lifts to (E, P) + we're done. \square

(Note: The schemes $\mathcal{Y}_i(N)/\mathbb{Z}$ are very
rarely smooth; this was true for $N=5$
essentially by accident.)

§ 3.5 A complex-analytic digestion

Let $\Lambda \subset \mathbb{C}$ be a lattice.

Def 3.5.1 The Weierstrass \wp -function

$\wp_\Lambda(z)$ is the unique holo. fn

$$\mathbb{C}/\Lambda \rightarrow \mathbb{P}^1(\mathbb{C})$$

$$\text{st } \wp_\Lambda(z) = \frac{1}{z^2} + \mathcal{O}(1)$$

at $z=0$, and $\wp_\Lambda(z)$ holomorphic away from Λ .

(the x -coordinate maps to differential dz .)

Machinery of prop 3.3.2 $\Rightarrow \mathbb{C}/\Lambda$ is
isomorphic to

$$Y^2 = X^3 - g_4 X - g_6$$

$$\text{via } z \mapsto \left(\wp(z), \frac{1}{z} \wp'(z) \right)$$

where g_4, g_6 are constants depending on Λ .

If $\Lambda = \Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$, the g_4 and g_6 are
constant multiples of Eisenstein series E_4, E_6 .
(holo. on $\text{bas of } \tau$)

Prop 3.5.2 Let $\mathcal{U} = \{ \tau, z \in \mathbb{H} \times \mathbb{C} \mid$
 $z, 2z, 3z \notin \Lambda_\tau \}$

Then \exists holomorphic map
 $\mathcal{U} \xrightarrow{(\alpha, \beta)} \mathbb{C}^2 = \{ \alpha, \beta \mid \Delta(\alpha, \beta) = 0 \}$

such that

$$E(\alpha(\tau, z), \beta(\tau z)) \cong \mathbb{C} / \Lambda_\tau$$

$$(0, 0) \longleftrightarrow z \bmod \Lambda_\tau.$$

Proof Start from the pair

$$\left(Y^2 = X^3 - g_4 X - g_6, \left(\wp(\tau z), -\frac{1}{i} \wp'(\tau z) \right) \right)$$

g_4, g_6 correspond to Λ_τ

Manipulate as in Prop 3.3.4 to put this in Tate normal form. All coeffs. of rescalings/translations are holomorphic on U as fns of (τ, z) . + hence so are resulting α, β . \square

Corollary 3.5.3

$Y_1(N)_{\mathbb{Z}[\frac{1}{N}]}(\mathbb{C})$ is isomorphic as a Riemann surface to $\Gamma_1(N) \backslash \mathcal{H}$.