# Modular Curves (TCC) Problem Sheet 1

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This is the first of 3 problem sheets, which will be distributed after lectures 3, 5, and 7 of the course. This problem sheet will be marked out of a total of 25; the number of marks available for each question is indicated.

Work should be submitted, on paper or by email, on or before Friday 14th February.

Throughout this sheet a *level* should be understood to mean a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

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1. [2 points] Let $X$ be a topological space and $G$ a group acting on $X$ by homeomorphisms (i.e. for any $g \in G$, the map $X \to X$ given by $g$ is a homeomorphism). Equip $G\backslash X$ with the quotient topology. Show that the image of an open subset of $X$ is an open subset of $G\backslash X$. Is the same statement true with 'open' replaced by 'closed'? Give a proof or counterexample as appropriate.

2. [3 points] Let $\Gamma$ be a level and let $\gamma_1, \dots, \gamma_r$ be such that $\mathrm{SL}_2 \mathbb{Z} = \bigsqcup_j \Gamma\gamma_j$. Show that $[\gamma_j i] \in Y(\Gamma)$ is elliptic if and only if $\gamma_j \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma_j^{-1} \in \Gamma$. Hence show that $Y_0(p)$, for $p$ an odd prime, has either 0 or 2 elliptic points of order 2 depending on the congruence class of $p$ modulo 4.

3. [2 points] Calculate the genus of $X_0(17)$. (You may assume an analogue of the preceding question for points of order 3 as long as it is stated clearly.)

4. [1 point] Does there exist a level $\Gamma$ which is neat (i.e. $\Gamma$ has no elements of finite order and all cusps are regular), but $X(\Gamma)$ has genus 0?

5. [3 points] Let $\Gamma = \Gamma_0(5)$. Show that $S_4(\Gamma)$ is one-dimensional, and that if $F$ is a basis vector of $S_4(\Gamma)$, then multiplication by $F$ is an isomorphism $M_k(\Gamma) \to S_{k+4}(\Gamma)$ for all $k \in \mathbb{Z}$.

6. [3 points] Let $X$ be a compact Riemann surface, $\mathcal{L}$ an invertible sheaf on $X$, and $P_1, \dots, P_n$ any finite set of points on $X$. Show that there exists a meromorphic section of $\mathcal{L}$ which is holomorphic and non-vanishing at all the $P_j$.

7. [3 points] Let $f : X \to Y$ be a non-constant morphism of Riemann surfaces. Let $P \in X$ and $Q = f(P) \in Y$. Show that if $\omega$ is a differential on $Y$ which is holomorphic and nonvanishing at $Q$, then the pullback $f^*\omega$ vanishes to order $e_P(f) - 1$ at $P$.

   Hence deduce the Riemann-Hurwitz formula, assuming that the sheaf of holomorphic differentials on a compact Riemann surface of genus $g$ has degree $2g - 2$.

8. [4 points] Let $X$ be a Riemann surface and $\omega$ a meromorphic differential on $X$. Define the *residue* of $\omega$ at a point $P \in X$. Show that if $X$ is compact, then $\sum_{P \in X} \mathrm{Res}_P(\omega) = 0$. (Hint: Use Stokes' theorem.)

9. [4 points] Let $\Gamma$ be a level, and let $\rho : \mathcal{H} \to \mathcal{H}$ be a homeomorphism such that $\rho(z) \in \Gamma z$ for every $z \in \mathcal{H}$. Show that $\rho(z) = \gamma z$ for a unique $\gamma \in \bar{\Gamma}$ (the image of $\Gamma$ in $\mathrm{PSL}_2 \mathbb{Z}$).

   Suppose $\bar{\Gamma}$ has no elements of finite order. Show that the fundamental group of $Y(\Gamma)$ (in the sense of algebraic topology) is isomorphic to $\bar{\Gamma}$.

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*The following questions are optional and wil not be assessed.*

10. Show that for any $\Gamma$ there exists an $R$ such that the graded ring $\bigoplus_{k\geq 0} M_k(\Gamma)$ is generated by forms of weight $\leq R$. What is the best bound you can find for $R$?
11. Does there exist a proper finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with only one cusp?
12. Show that there are infinitely many levels such that $X(\Gamma)$ has genus 1. (Hint: Consider subgroups of $\Gamma_0(11)$.)